BMS-Summer School **Probabilistic and statistical methods for networks** 21.8.-1.9.2017

Selfish routing in networks

Max Klimm

special thanks to Philipp Warode

Main questions

- users of these networks make uncoordinated and selfish decisions

- Description:

  What kind of usage pattern emerges?

- Computation:

  Can the pattern be computed efficiently?

- Efficiency:

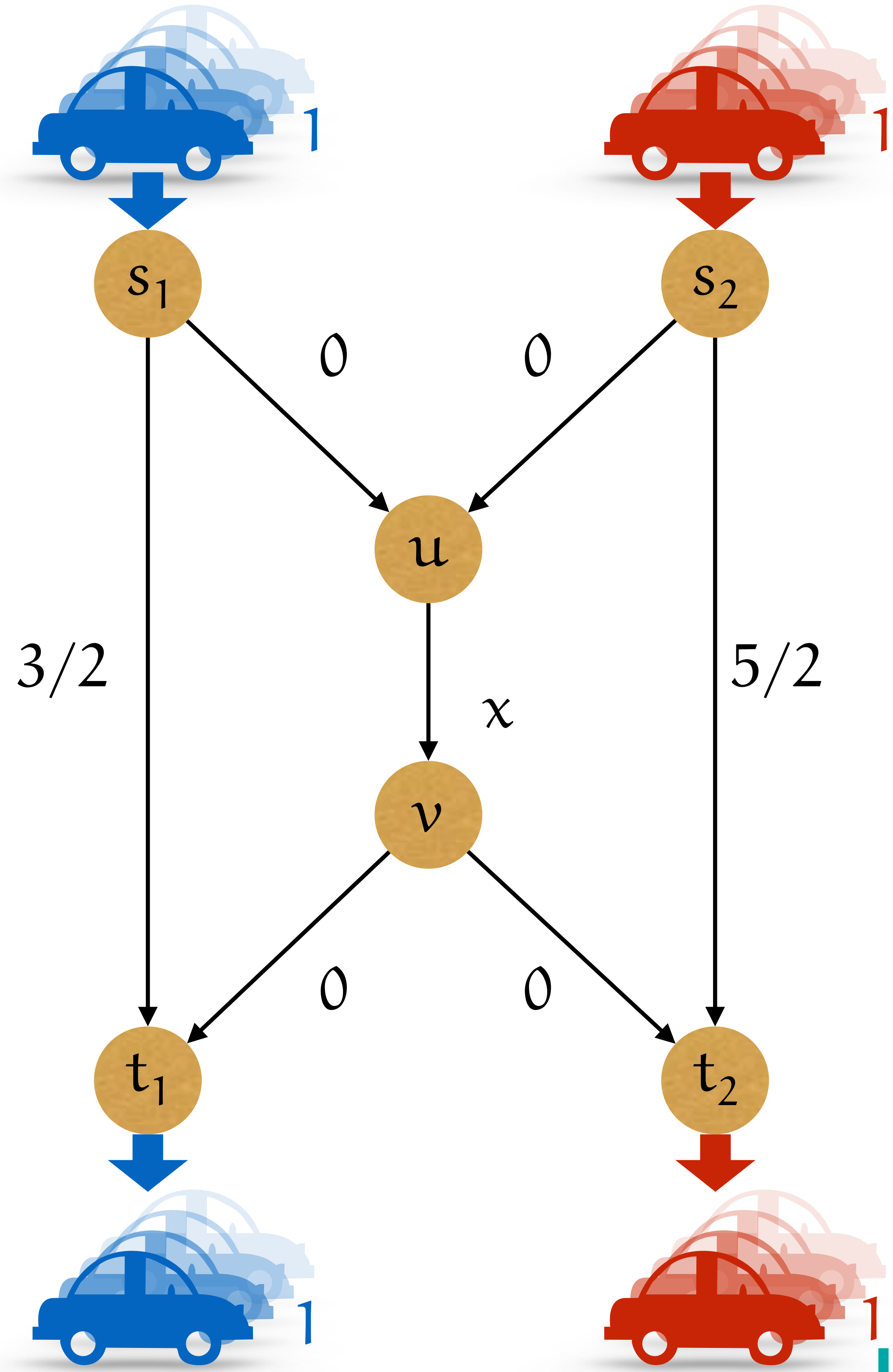
  How efficient is this usage compared to the optimum?

Equilibrium flows

Introduction

Introduction to selfish flows

- two unit size populations of drivers
  - blue: going from s_1 to t_1
  - red: going from s_2 to t_2
- each driver has two path choices
  - blue: $s_1 \rightarrow t_1$ or $s_1 \rightarrow u \rightarrow v \rightarrow t_1$
  - red: $s_2 \rightarrow t_2$ or $s_2 \rightarrow u \rightarrow v \rightarrow t_2$
- travel time along an edge depends on the total traffic on that edge
- each driver is interested in minimizing its own travel time



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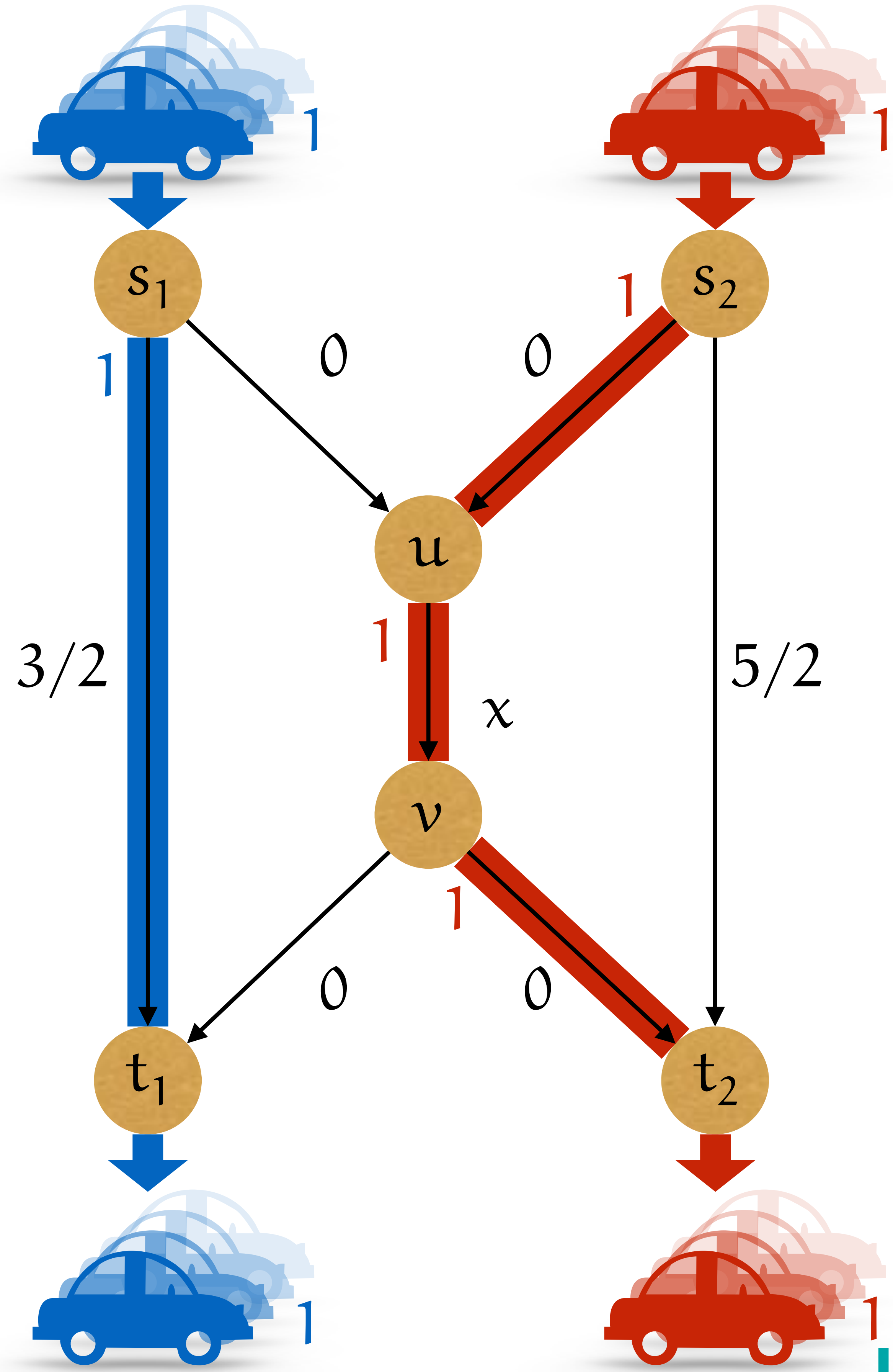

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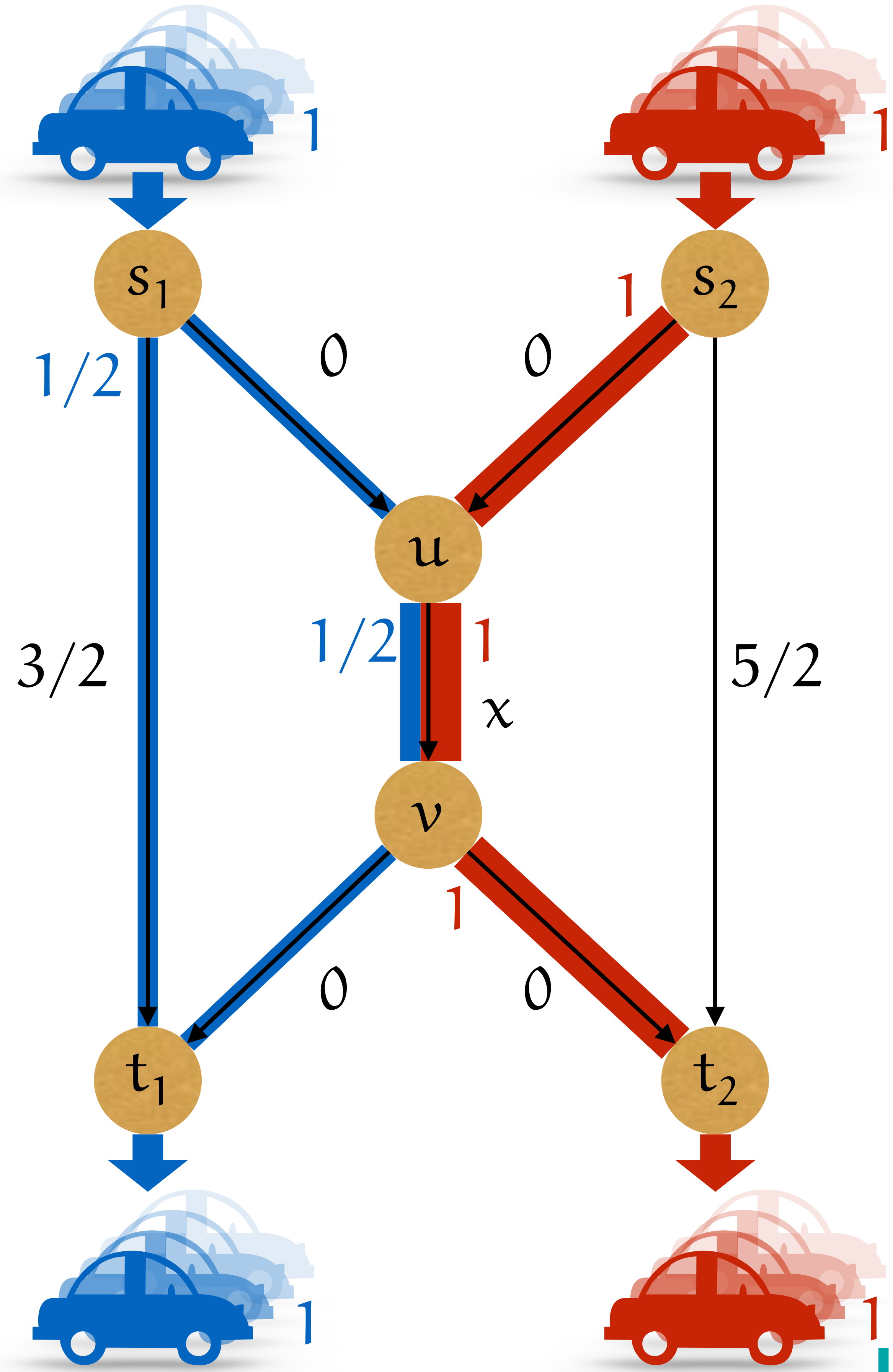
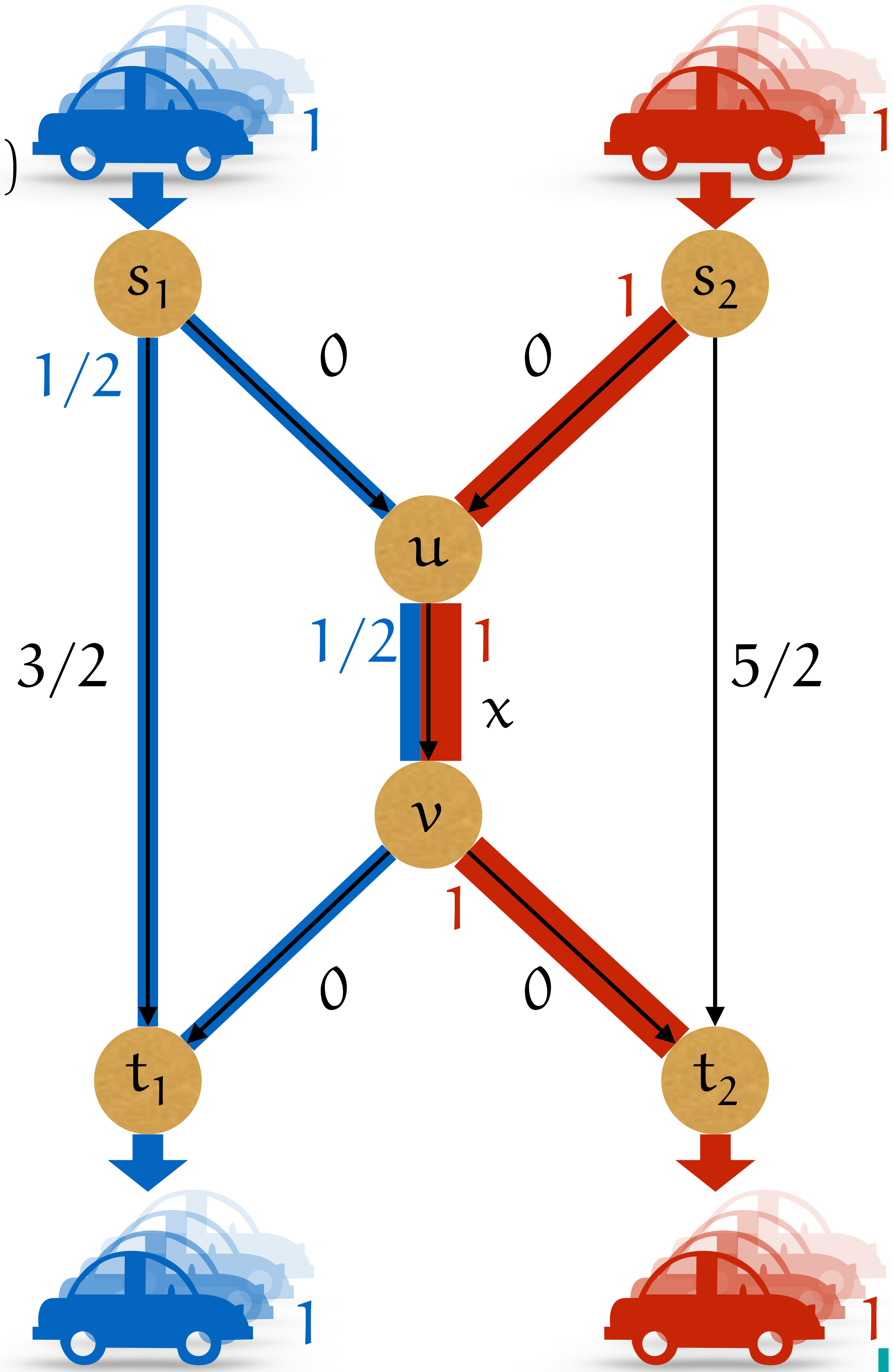


Formal model

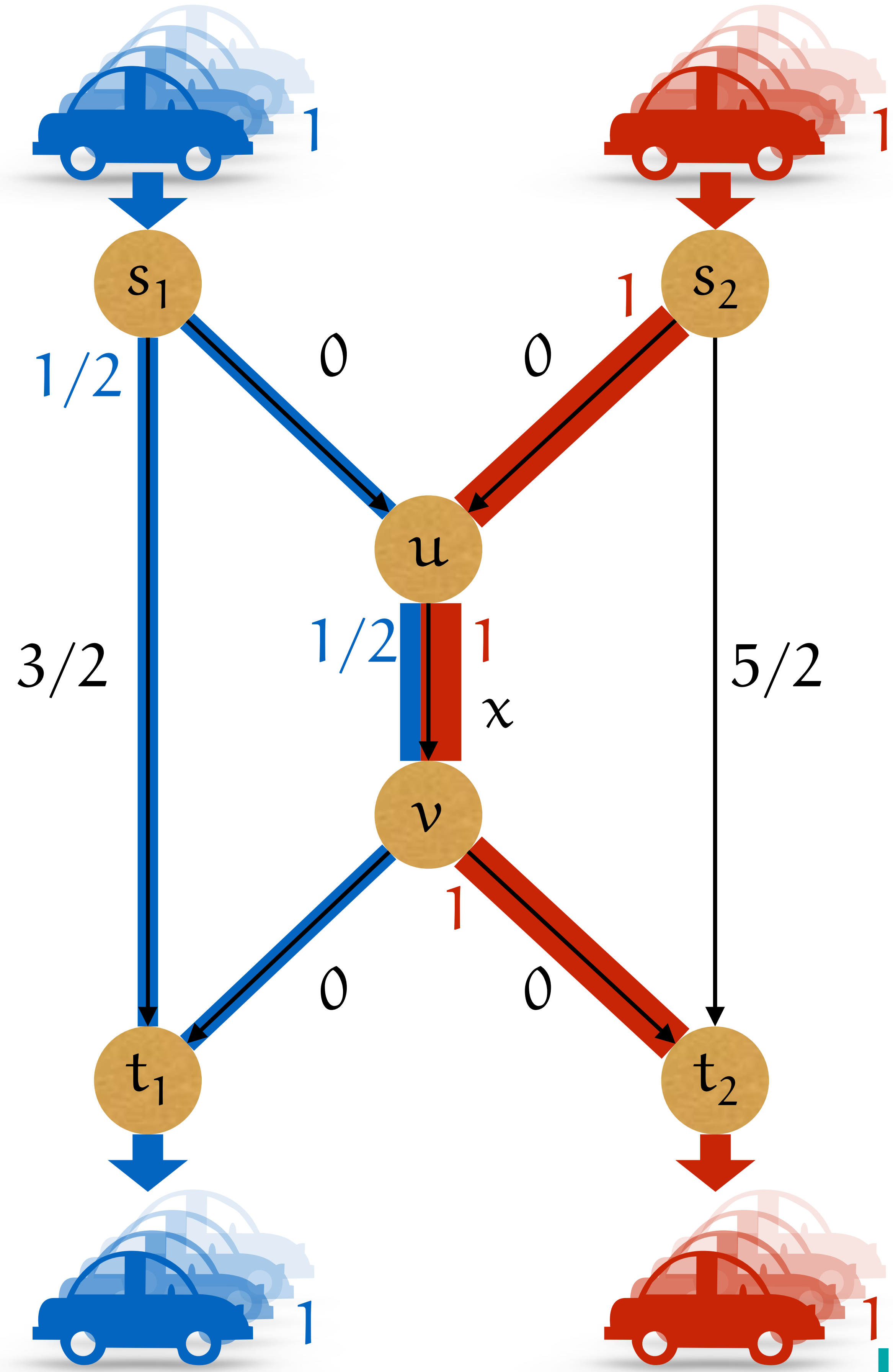
- directed or undirected graph $G = (V, E)$
  - finite set of vertices V
  - set of edges $E \subseteq V \times V$
- cost function $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ for $e \in E$
  - non-decreasing
  - continuous
  - (convex)
- finite set K of commodities (s_i, t_i, d_i)
  - origin vertex $s_i \in V$
  - destination vertex $t_i \in V$
  - demand $d_i \in \mathbb{R}_+$



Flows

- $\mathcal{P}_i$ = set of paths from s_i to t_i

Definition — Flow (Path formulation)
Collection of functions $f_i : \mathcal{P}_i \in \mathbb{R}_+$
with $\sum_{P \in \mathcal{P}_i} f_i(P) = d_i$ for all $i \in K$.



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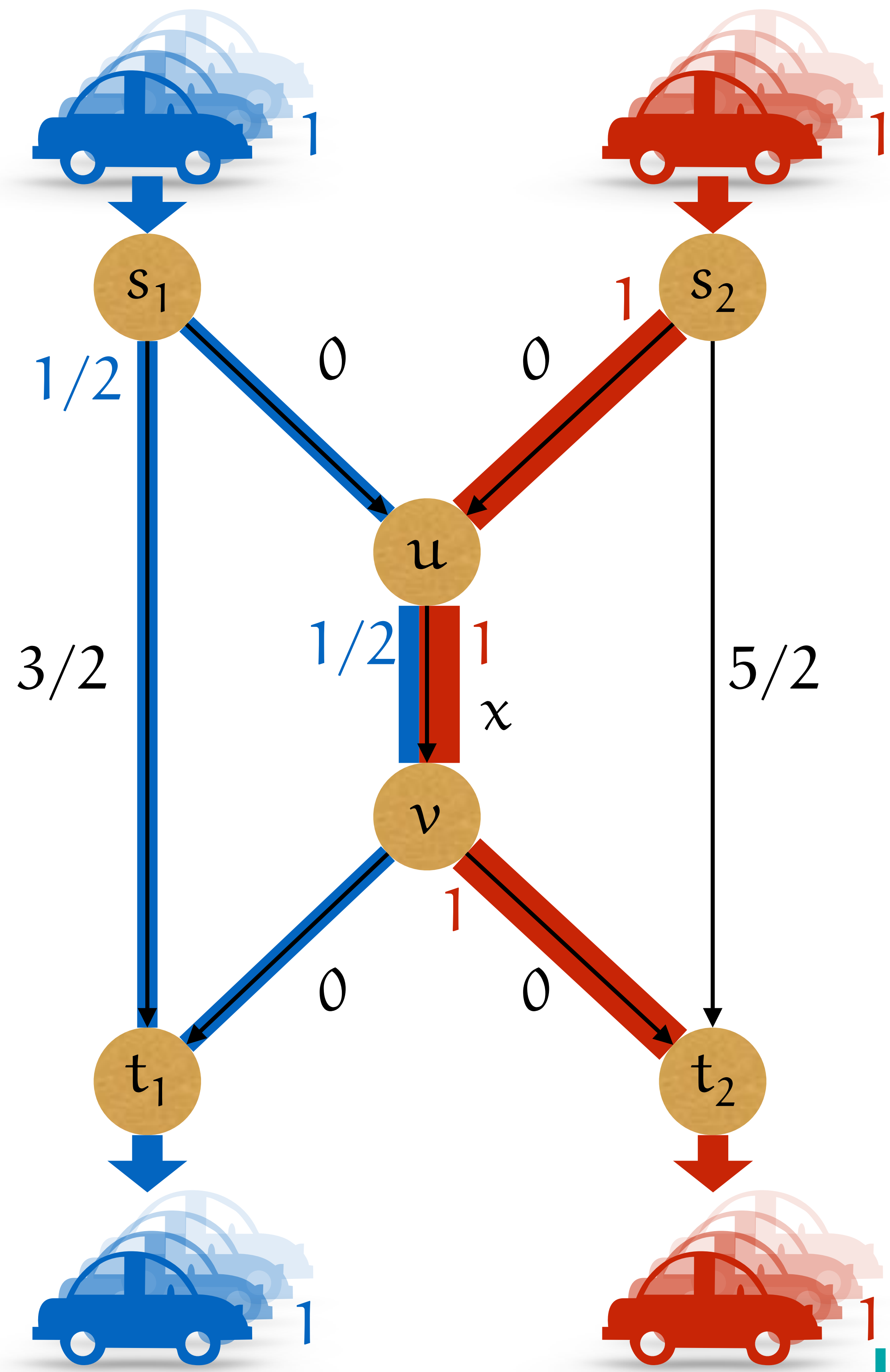

Definition — Flow (Edge formulation)

Collection of functions $f_i : E \to \mathbb{R}_+$ satisfying flow conservation laws:

$$\sum_{e \in \delta^+(v)} f_i(e) = \sum_{e \in \delta^-(v)} f_i(e) \ \forall v \in V \setminus \{s_i, t_i\}$$

$$\sum_{e \in \delta^+(s_i)} f_i(e) - \sum_{e \in \delta^-(s_i)} f_i(e) = d_i$$

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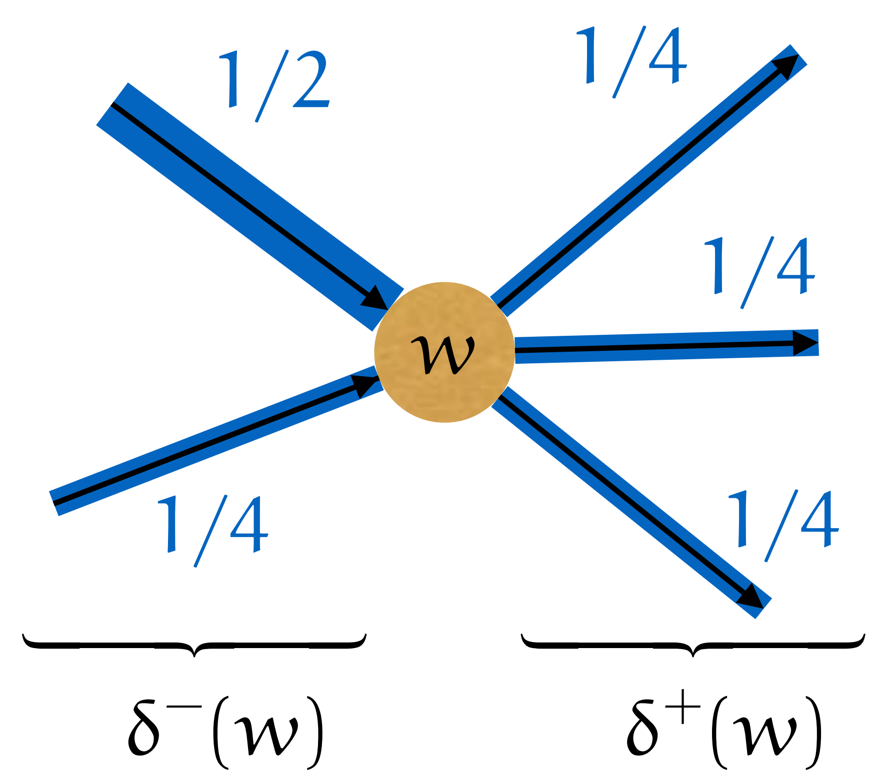

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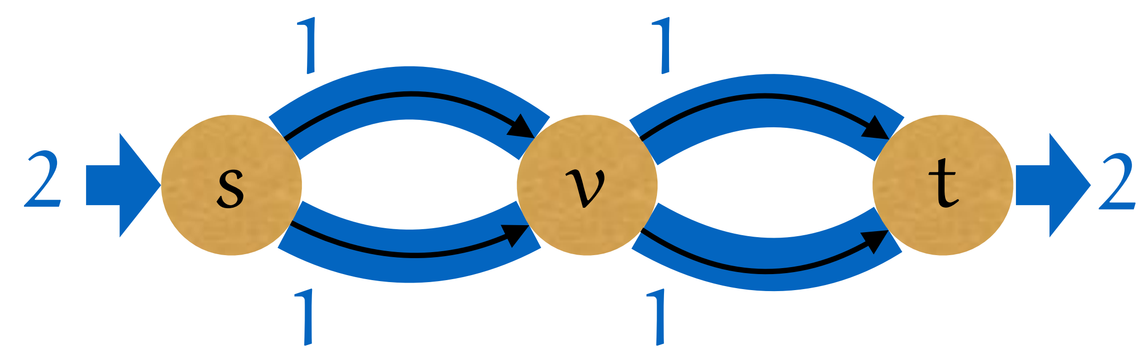
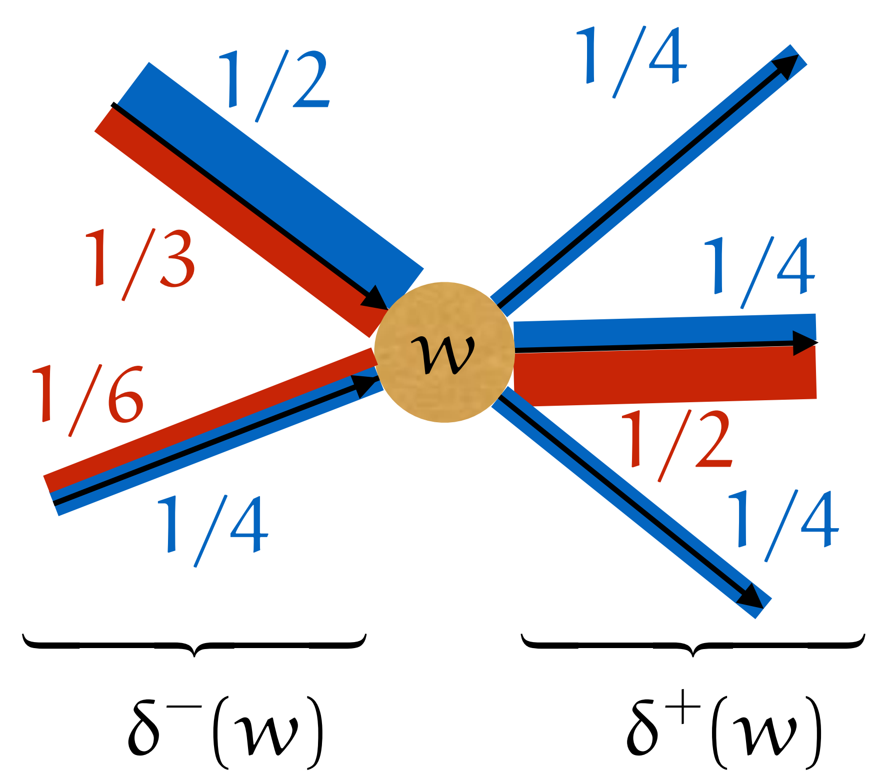


$f_i(e) = \sum_{P \ni e} f_i(P)$

unique

not unique



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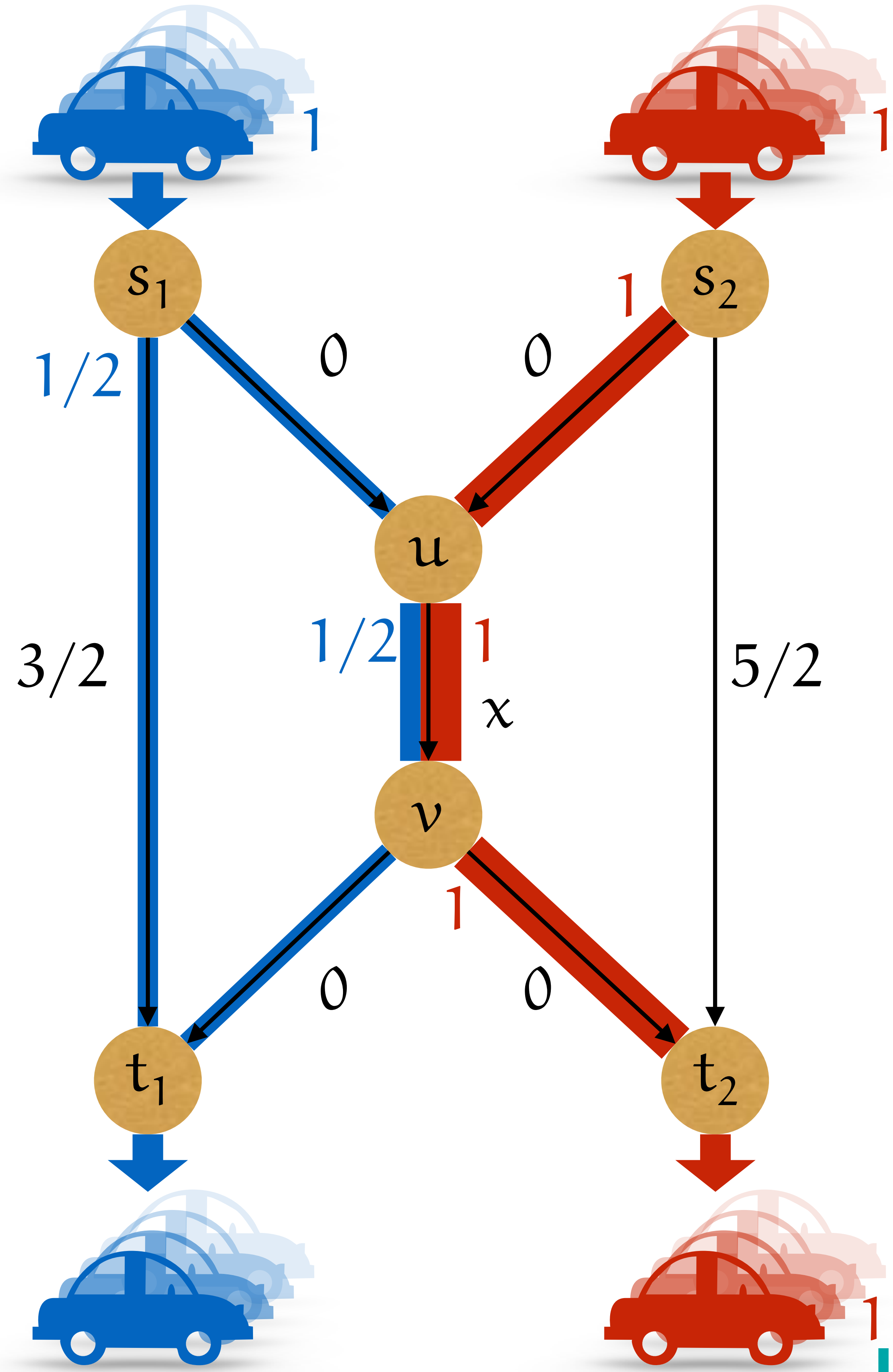
$\sum_{e \in \delta^+(s_i)} f_i(e) - \sum_{e \in \delta^-(s_i)} f_i(e) = d_i$

Equilibrium flows

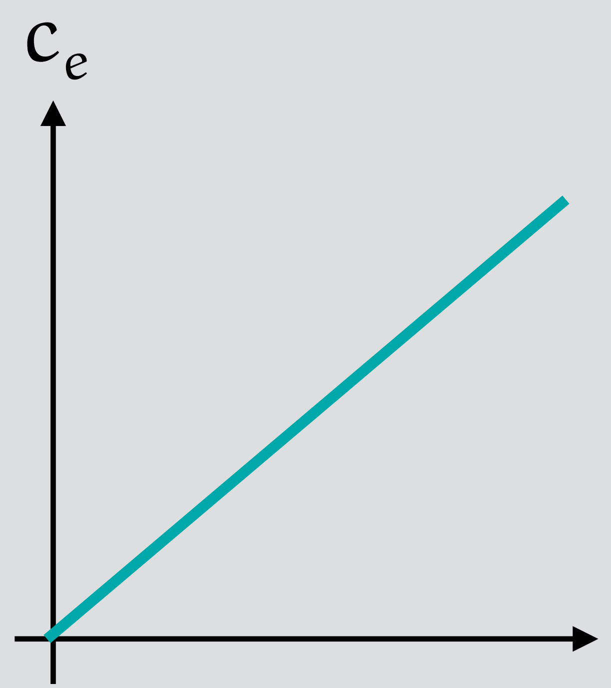
- "The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route." [Wardrop '52]

- $f(e) = \sum_{i \in K} f_i(e)$

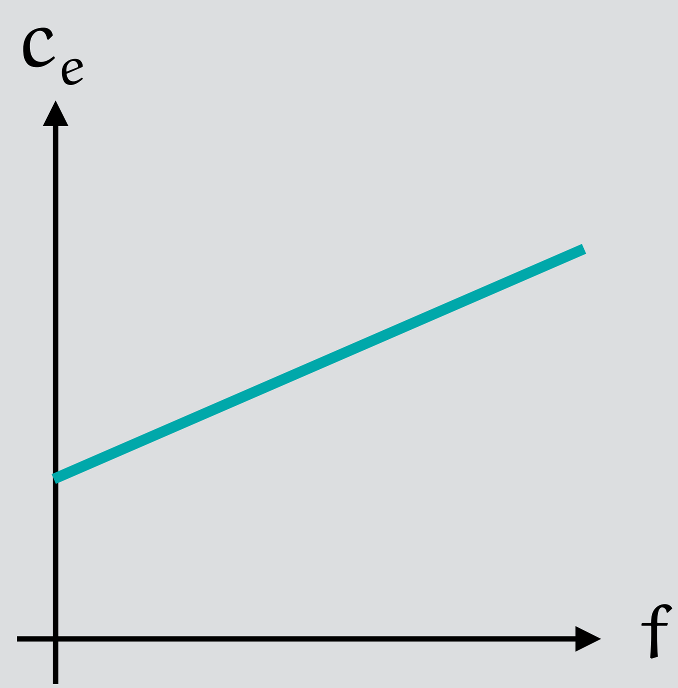
Definition — Wardrop equilibrium:
Path flow $f = (f_i)_{i \in K}$ with
$\sum_{e \in P} c_e(f(e)) \leqslant \sum_{e \in Q} c_e(f(e))$
for all $i \in K$, and $P, Q \in \mathcal{P}_i$ with $f_i(P) > 0$.



Notable cost functions

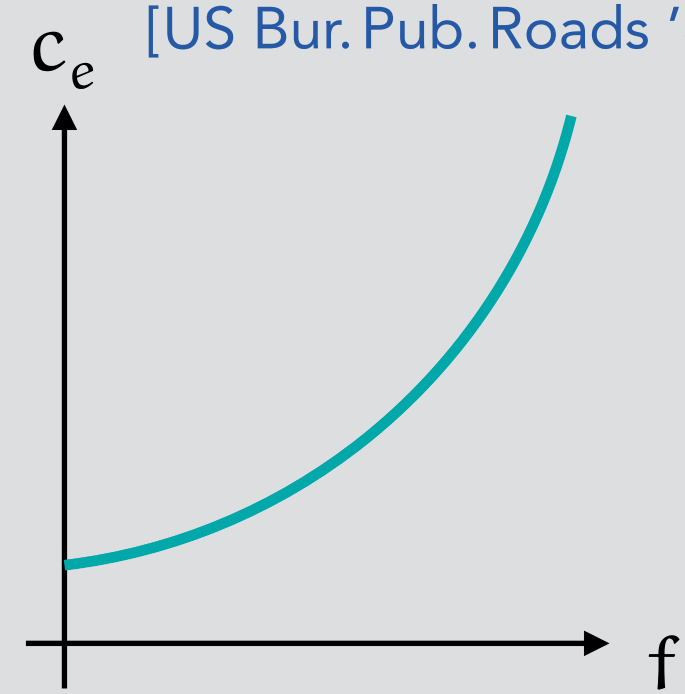


linear function
$c_e(x) = ax$

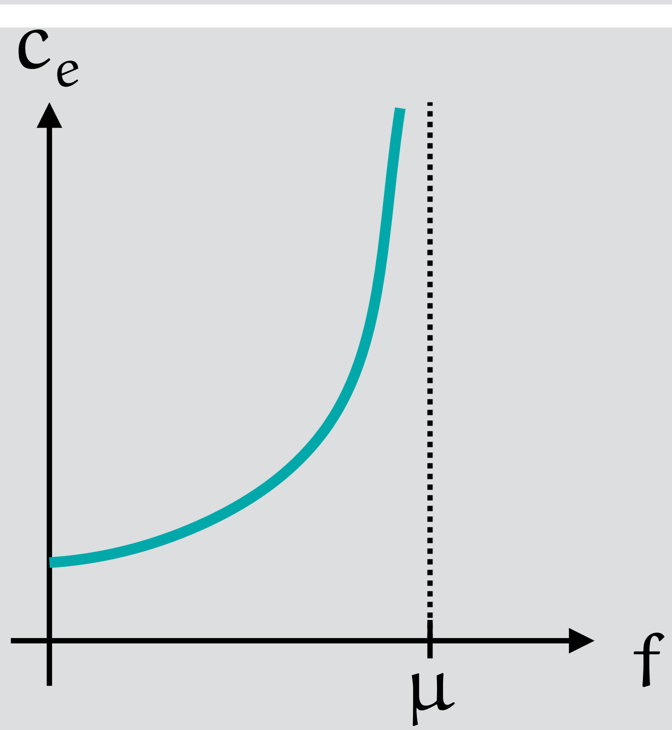


affine function
$c_e(x) = ax + b$

[US Bur. Pub. Roads '64]



BPR function
$c_e(x) = t(1+0.15(x/u)^4)$



MM1 function
$c_e(x) = 1/(\mu - x)$

- expected response time of single server
- service time exponentially distributed with parameter μ
- arrivals according to Poisson process at rate x

Equilibrium flows

Existence and uniqueness

Characterization of Wardrop equilibria

Theorem [Beckman et al. '56]

The following are equivalent:

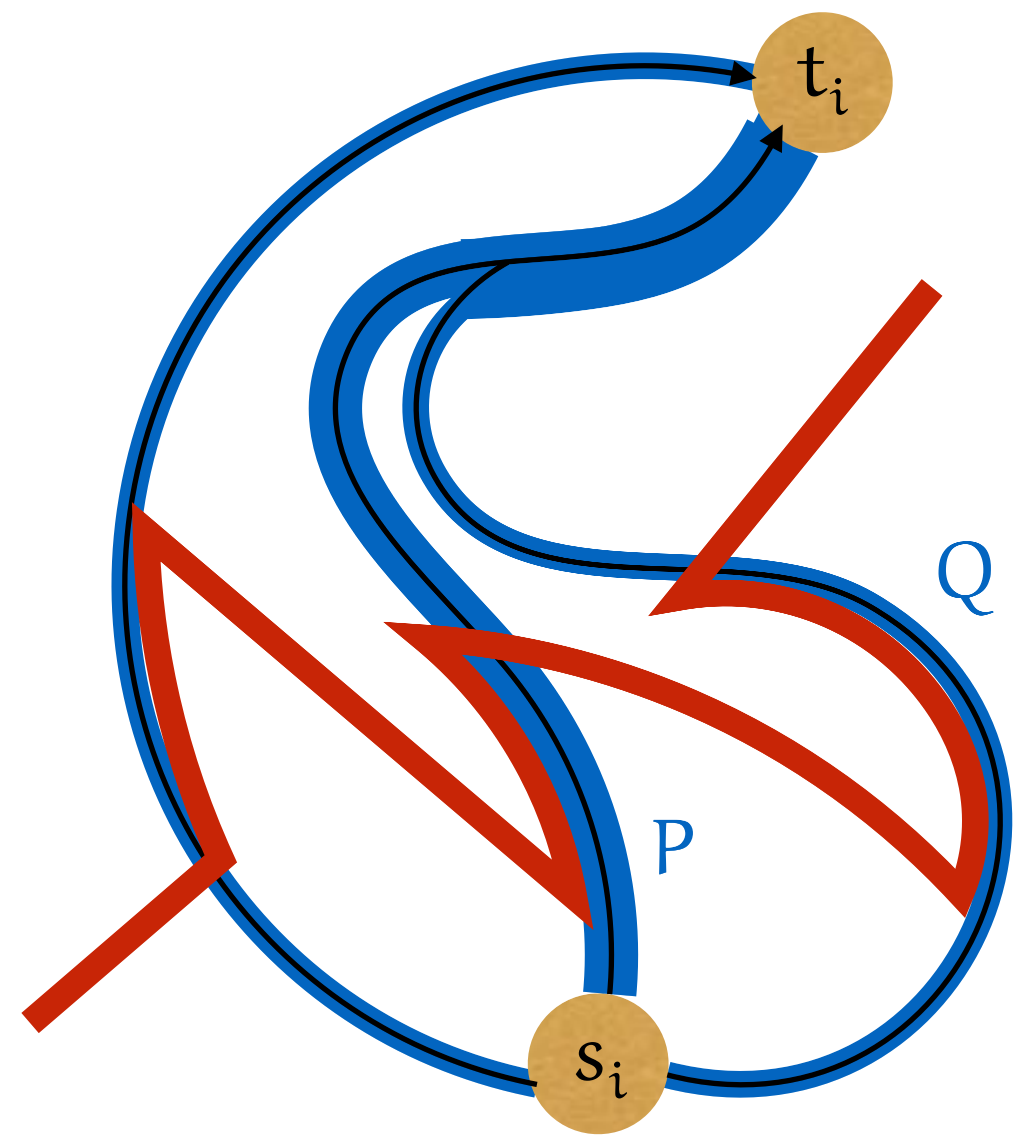
1. f is a Wardrop equilibrium.
2. f satisfies the variational inequality
   $\sum_{e \in E} c_e(f(e))\big(g(e) - f(e)\big) \geqslant 0 \quad \forall$ flows $g : E \to \mathbb{R}_+$.
3. f is an optimal solution to
   minimize $\sum_{e \in E} \int_0^{g(e)} c_e(t)\, dt$ s.t. $g : E \to \mathbb{R}_+$ is a flow .

- 3. yields an efficient algorithm as the minimization problem can be solved with convex optimization techniques

Proof of characterization

f WE ⇔ f satisfies (VI) $\sum_{e \in E} c_e(f(e))(g(e) - f(e)) \geqslant 0$

- "$\Leftarrow$"
  - let $i \in K$, and paths $P, Q \in \mathcal{P}_i$ with $\lambda = f_i(P) > 0$ be arbitrary
  - consider new flow f' with $f'(Q) = f(Q) + f(P)$ and $f'(P) = 0$



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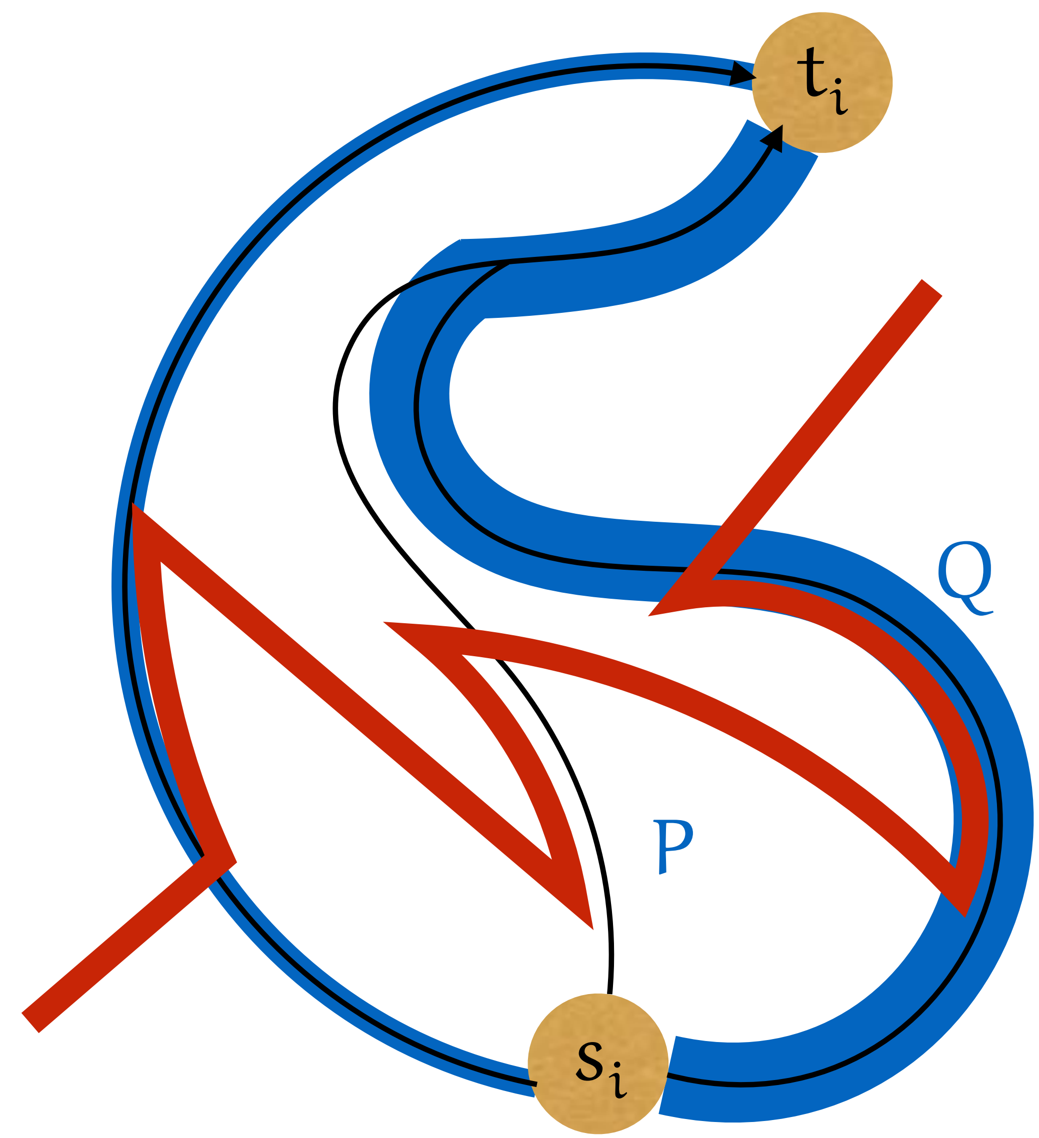
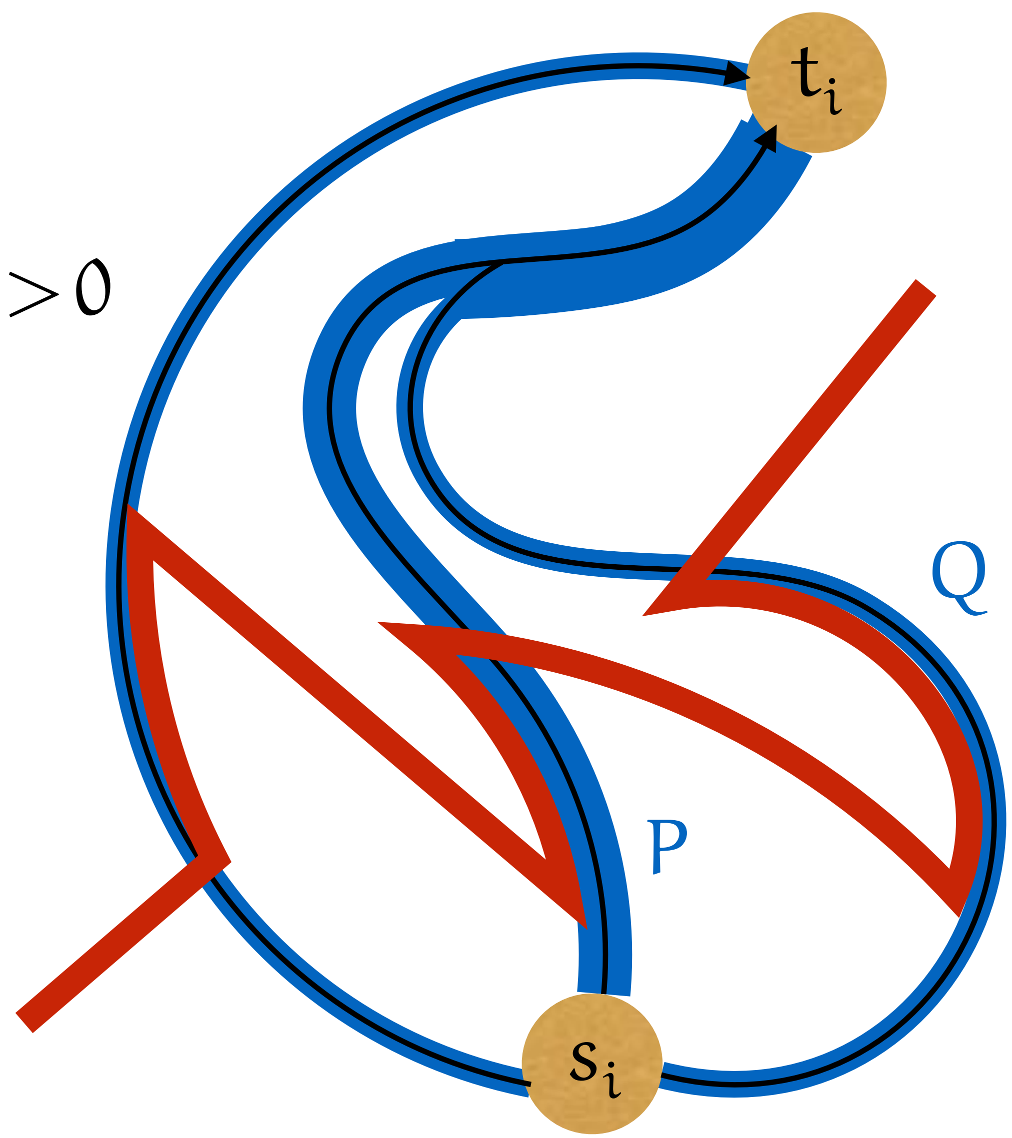
  - consider new flow f' with $f'(Q) = f(Q) + f(P)$ and $f'(P) = 0$
  - by (VI),
    $0 \leq \sum_{e\in E} c_e(f(e))(f'(e) - f(e))$
    $= \lambda \left(\sum_{e\in Q} c_e(f(e)) - \sum_{e\in P} c_e(f(e))\right)$
  - f is a WE



Proof of characterization

f WE $\Leftrightarrow$ f satisfies (VI) $\sum_{e\in E} c_e(f(e))(g(e) - f(e)) \geqslant 0$

- "$\Rightarrow$"
  - for a WE f, and $i \in K$,
    there are constants $k_i \in \mathbb{R}_+$ with
    $\sum_{e\in P} c_e(f(e)) = k_i$ for all $P \in \mathcal{P}_i$ with $f(P) > 0$
  - $$\begin{aligned} &\sum_{e\in E} c_e(f(e))f(e) \\ &= \sum_{i\in K} k_i\, d_i \\ &= \sum_{i\in K} \sum_{P\in\mathcal{P}_i} k_i\, g_i(P) \\ &\leqslant \sum_{i\in K} \sum_{P\in\mathcal{P}_i} g_i(P) \sum_{e\in P} c_e(f(e)) \\ &= \sum_{e\in E} c_e(f(e))g(e) \end{aligned}$$



Proof of characterization

f min. $\sum_{e\in E}\int_0^{g_e} c_e(t)\,dt \Leftrightarrow$ f satisfies (VI) $\sum_{e\in E} c_e(f(e))(g(e)-f(e)) \geqslant 0$

- here only "$\Leftarrow$"
- let $h(\mathbf{g}) = \sum_{e\in E} \int_0^{g(e)} c_e(t)\,dt$
- the optimization problem

  min. $h(\mathbf{g})$ s.t. $\mathbf{g}$ is a flow

  is convex on a convex domain
- first-order Taylor approximation in $\mathbf{f}$ gives

  $T_h(\mathbf{g};\mathbf{f}) = h(\mathbf{f}) + (\mathbf{g}-\mathbf{f})^T\,\nabla h(\mathbf{f})$
- $\qquad\qquad = h(\mathbf{f}) + \sum_{e\in E} c_e(f(e))\big(g(e) - f(e)\big)$
- so, when f satisfies (VI)

  $h(\mathbf{g}) \geqslant T_h(\mathbf{g};\mathbf{f})$ (by convexity)

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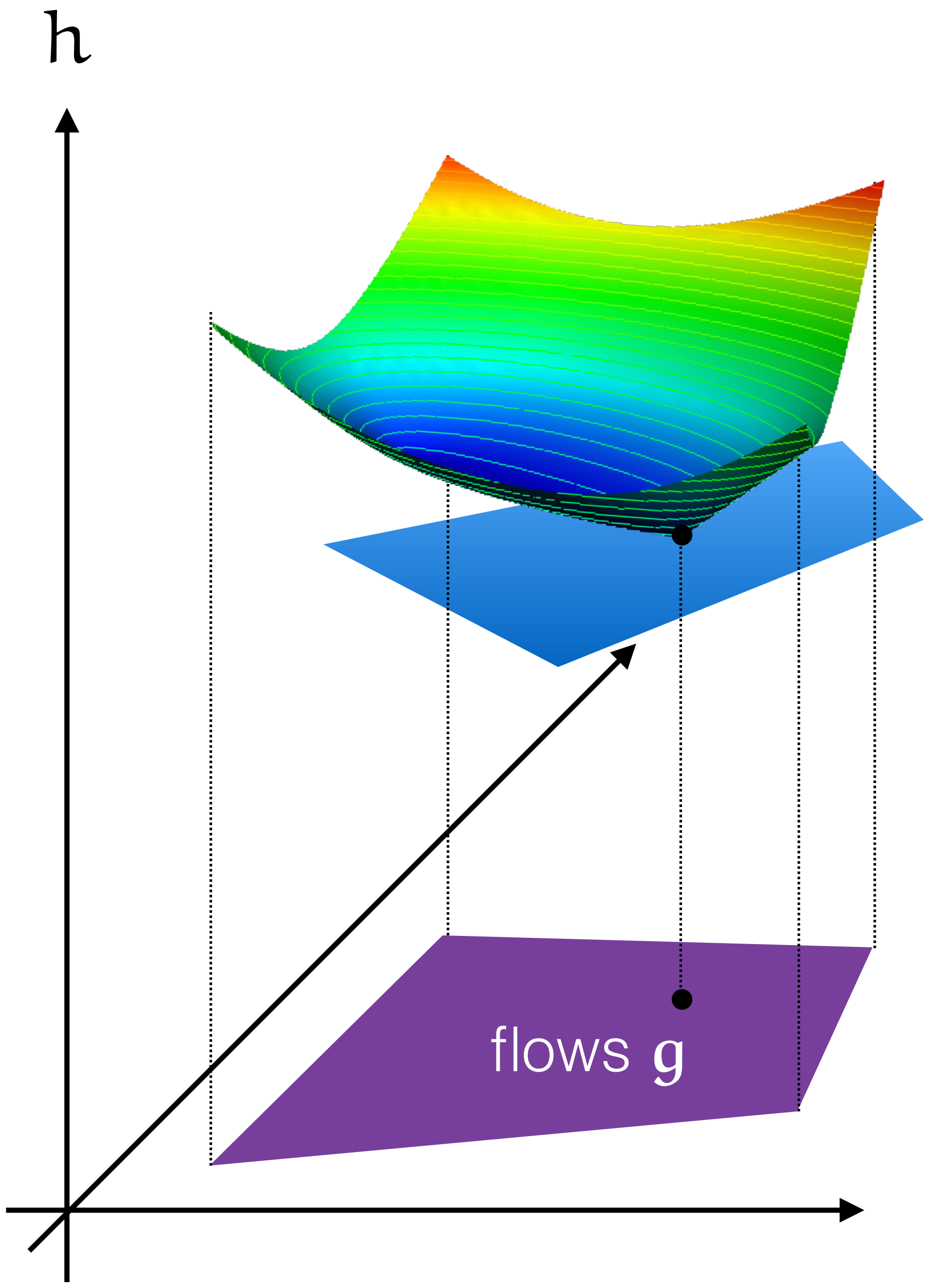


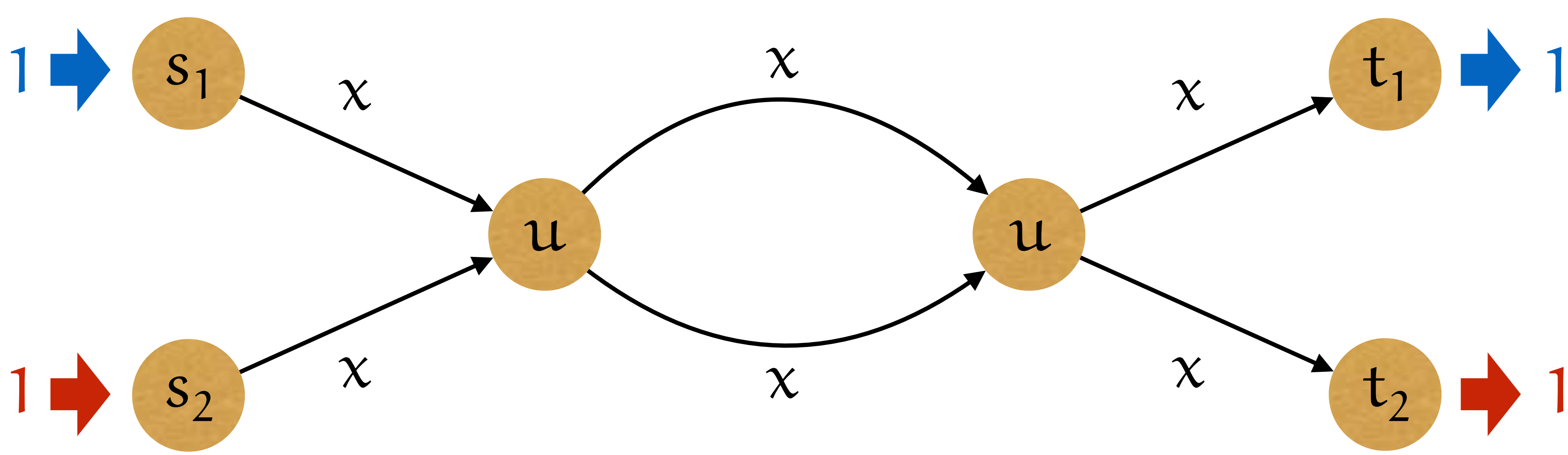
Uniqueness of equilibria

Corollary [Beckman et al. '56]

If cost functions are non-constant everywhere, the total edge flows $f_e = \sum_{i \in K} f_i(e)$ of all Wardrop equilibria f are unique.

- for non-constant functions, $h(g) = \sum_{e \in E} \int_0^{g(e)} c_e(t)\,dt$ is strictly convex
- unique minimum f

- path flow may not be unique though



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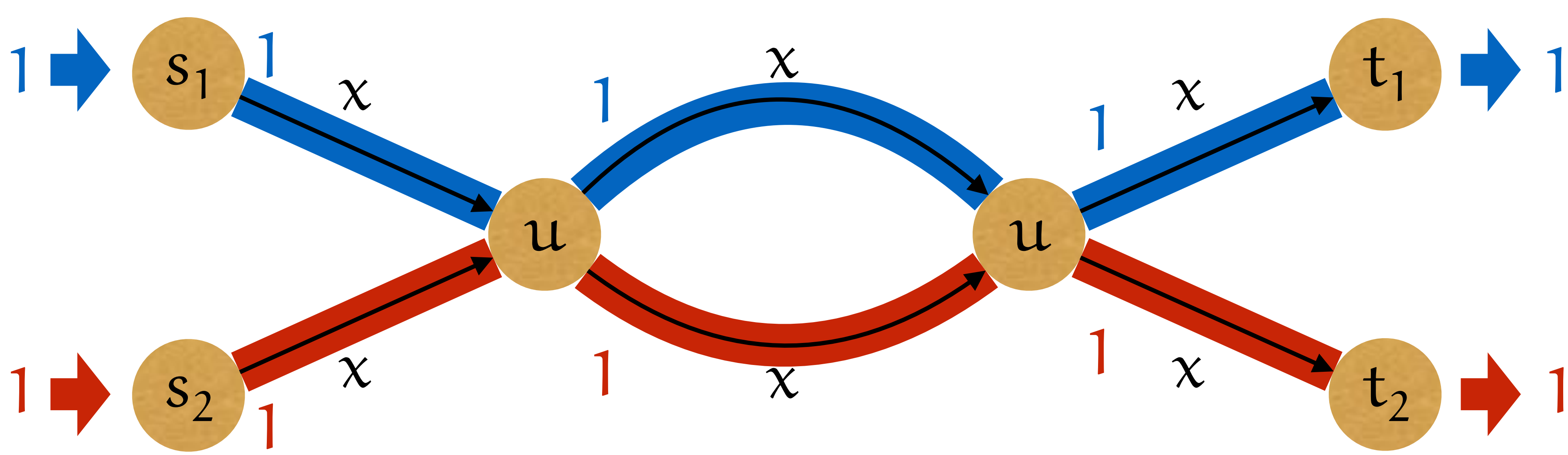

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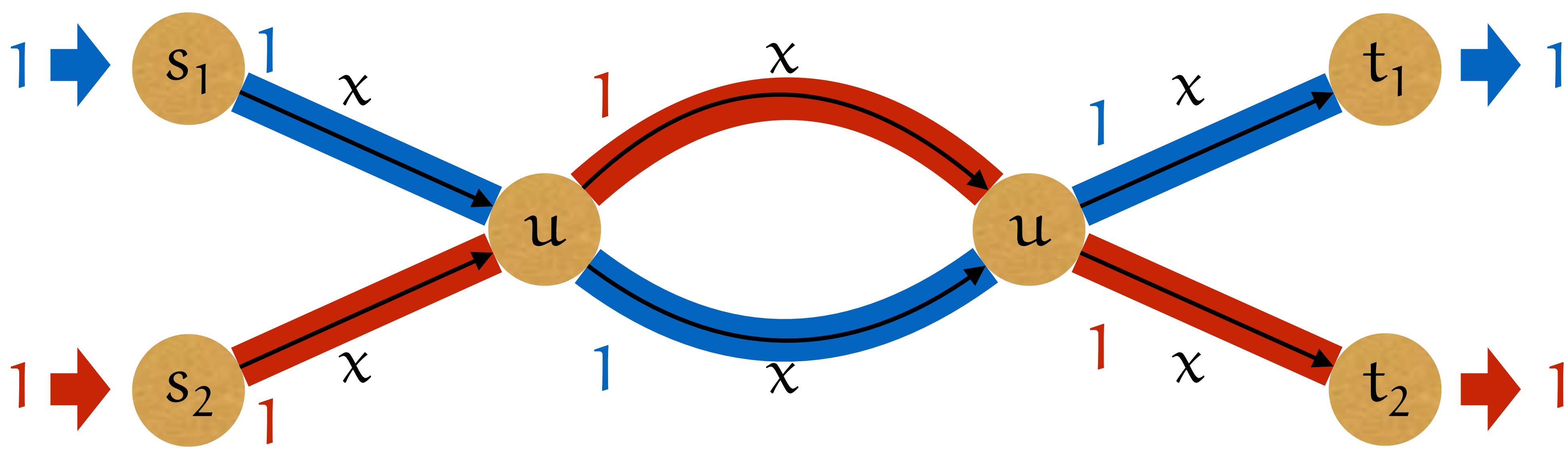

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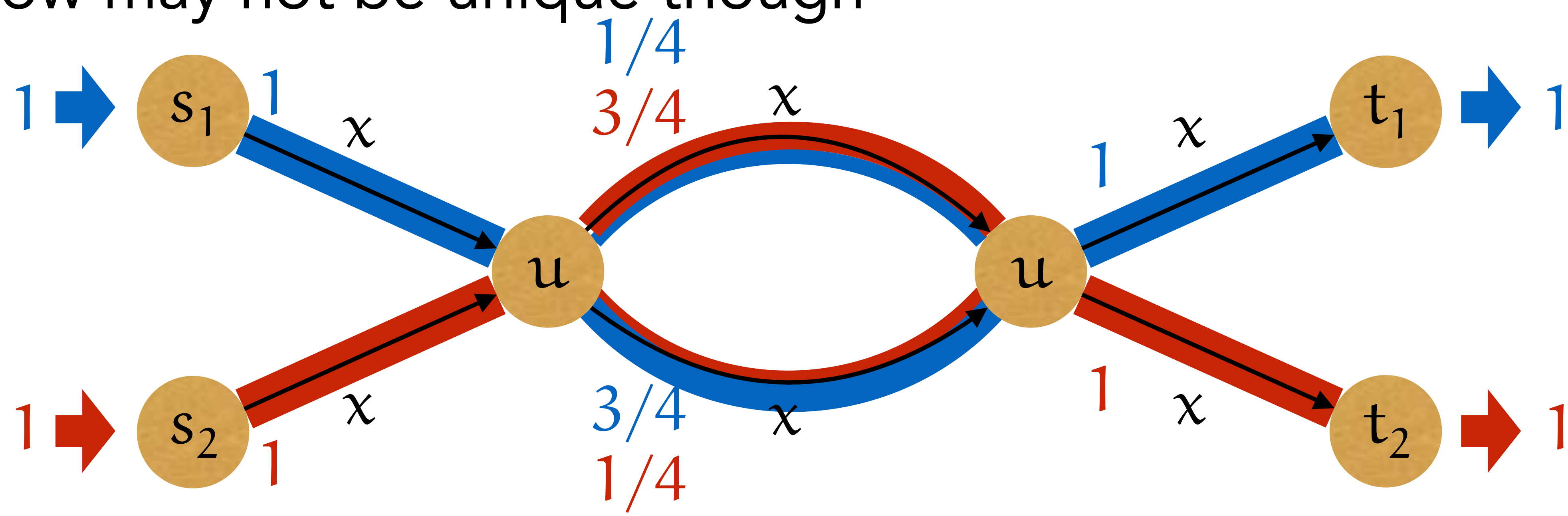
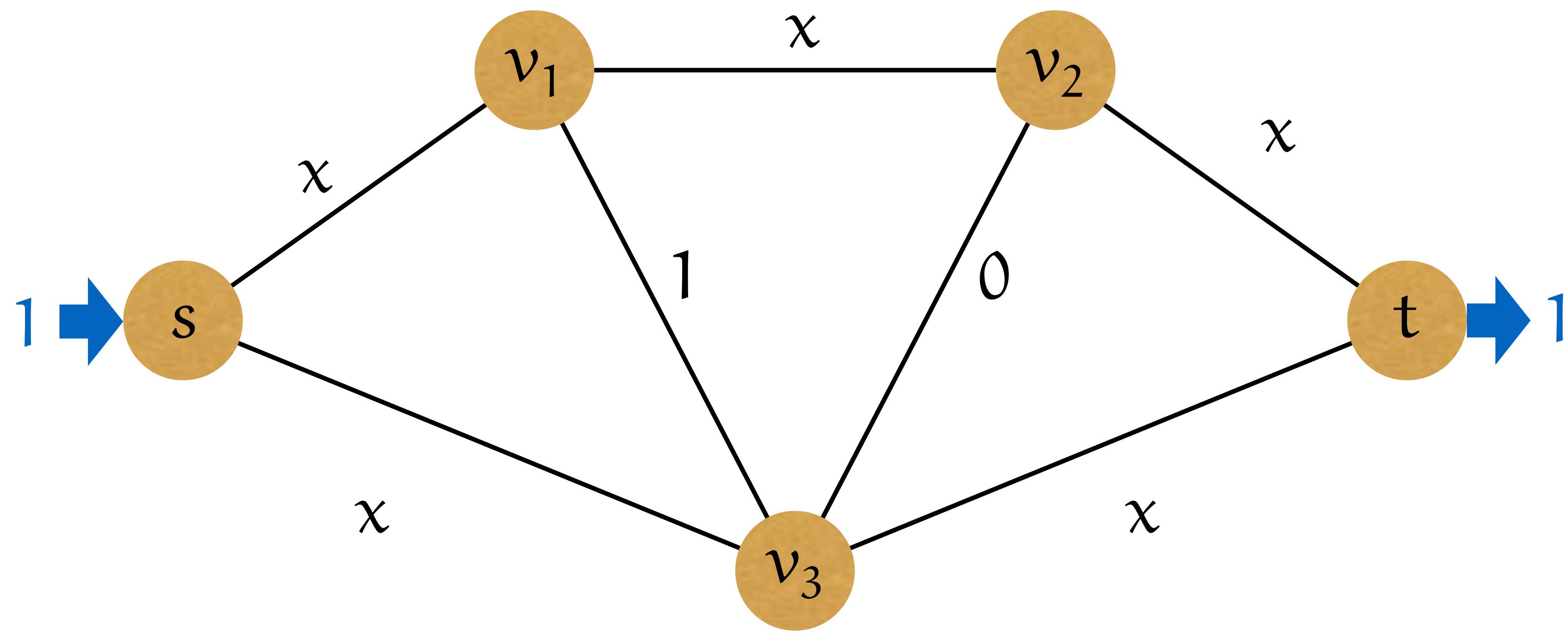


Equilibrium flows

Undirected single-commodity networks

Characterization of edge flows



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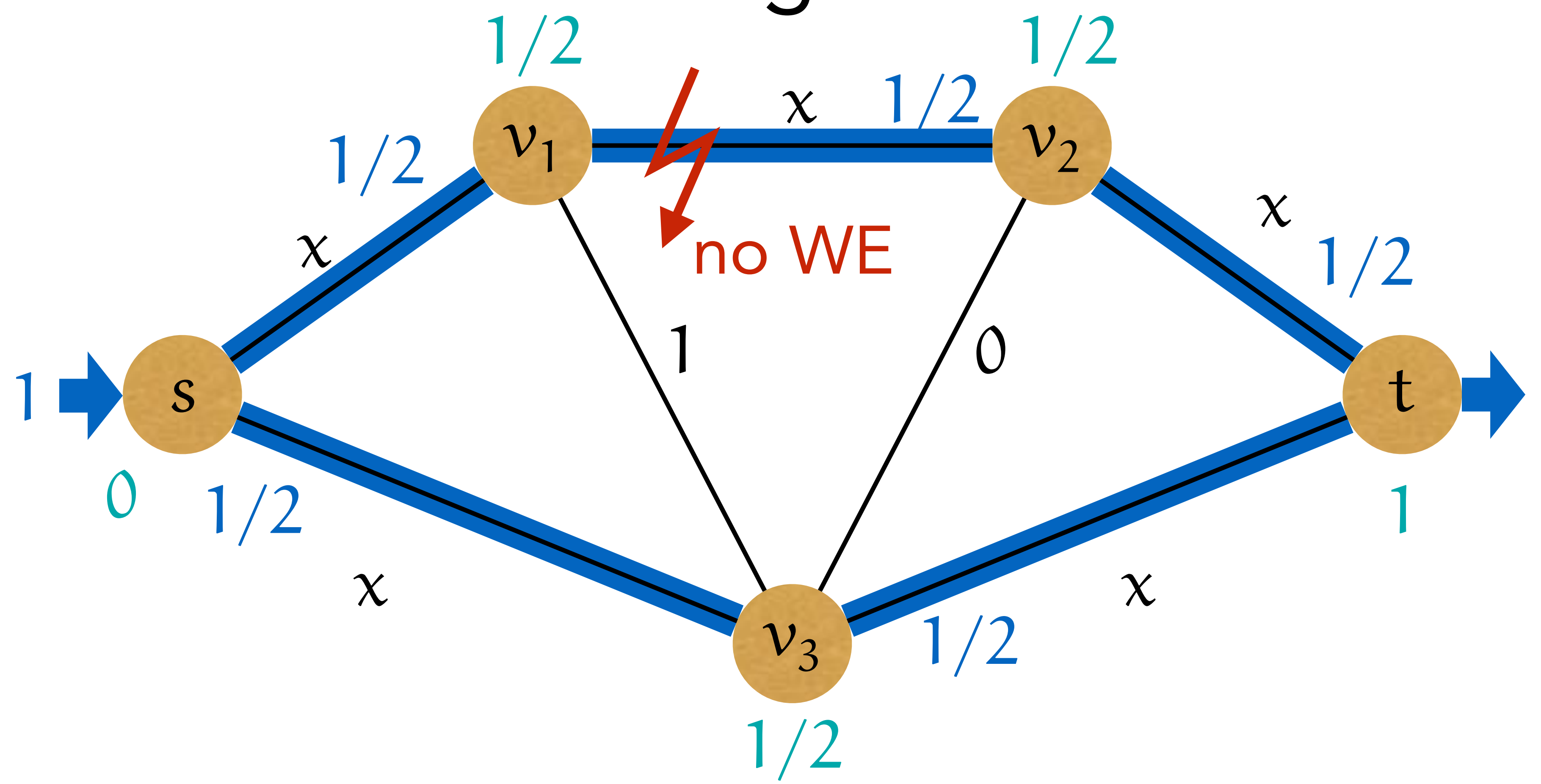
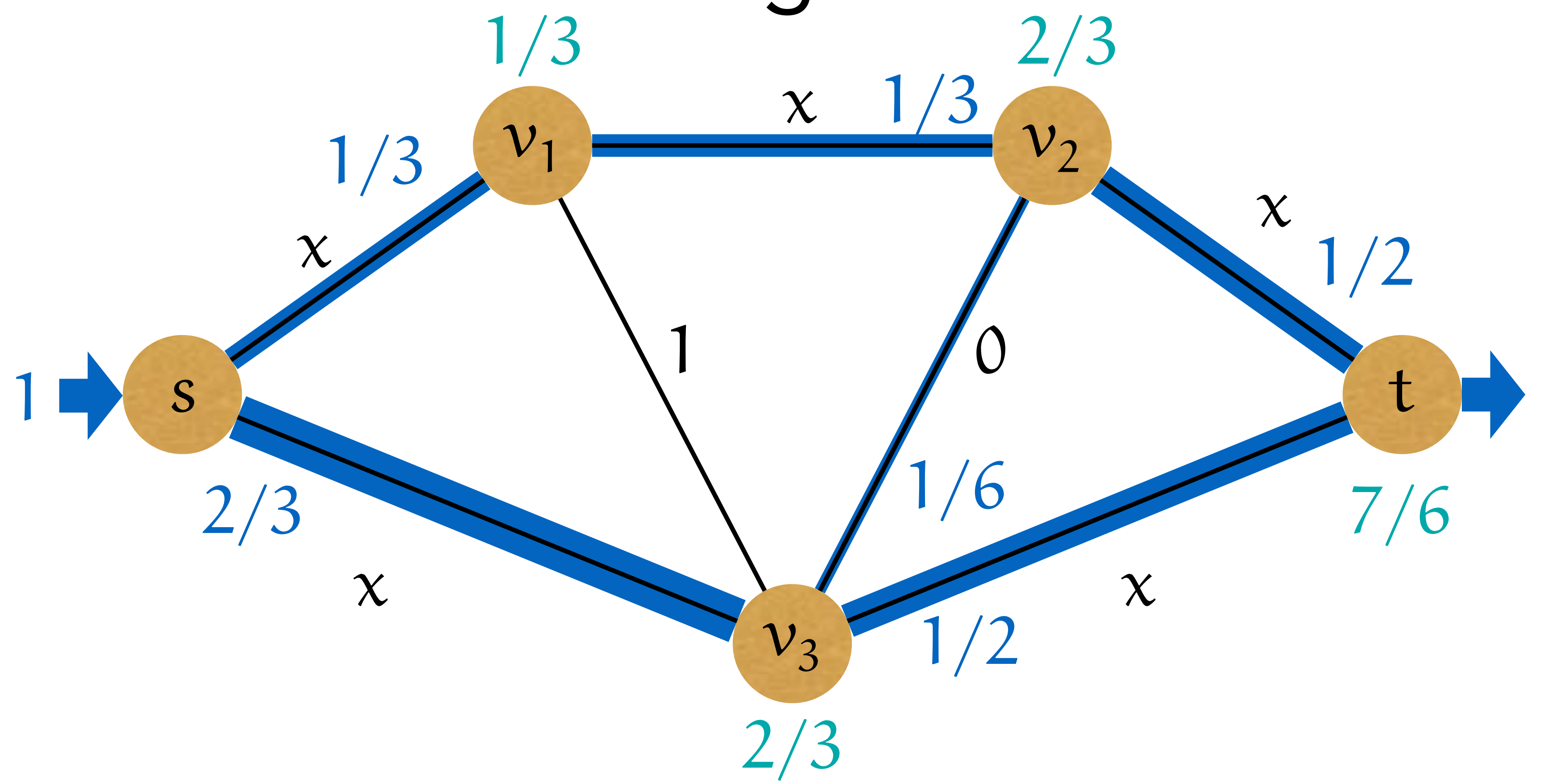


- for a fixed flow f, let $\pi(v)$ be the length of a shortest path from s to v (in terms of $c_e(f(e))$)
- $\pi(w) - \pi(v) \leqslant c_e(f(v,w))$ for every edge $(v,w) \in E$.

Lemma

f WE $\Leftrightarrow \pi(w) - \pi(v) = c_e(f(v,w))$ for all edges with $f(v,w) > 0$.

Characterization of edge flows



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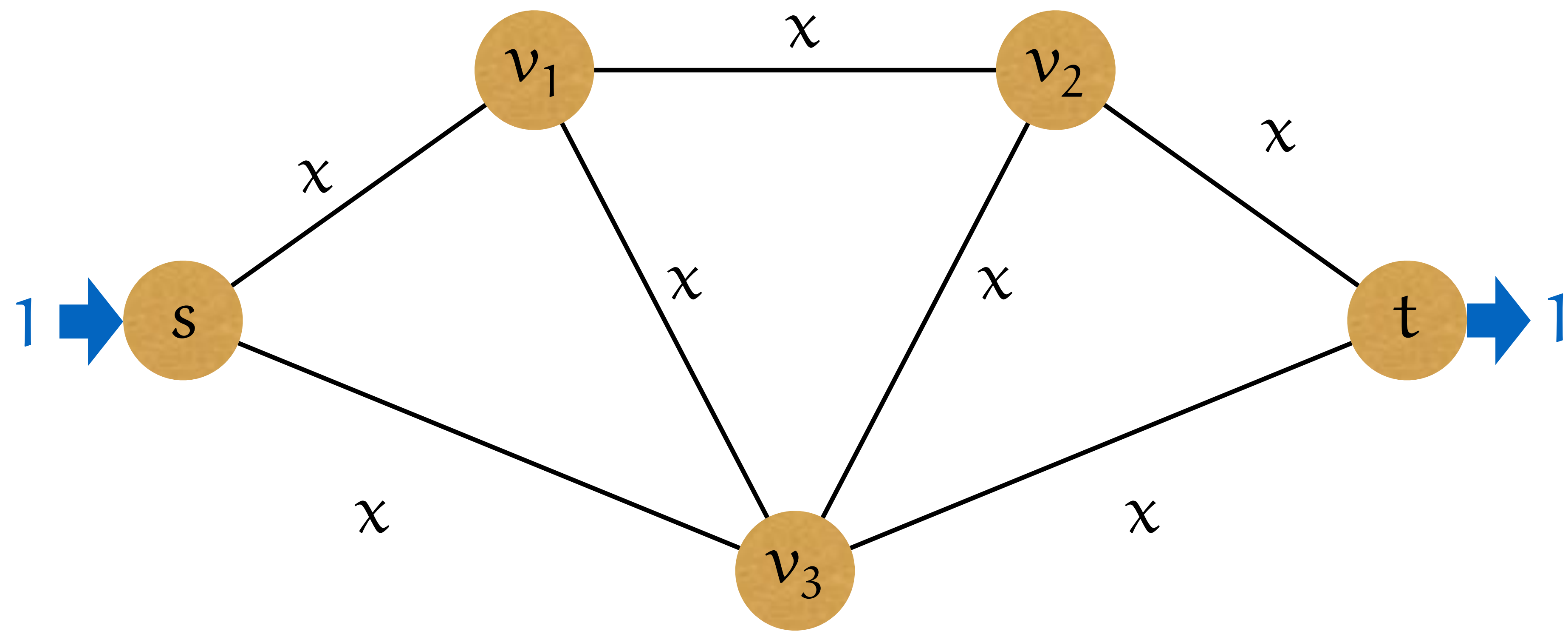

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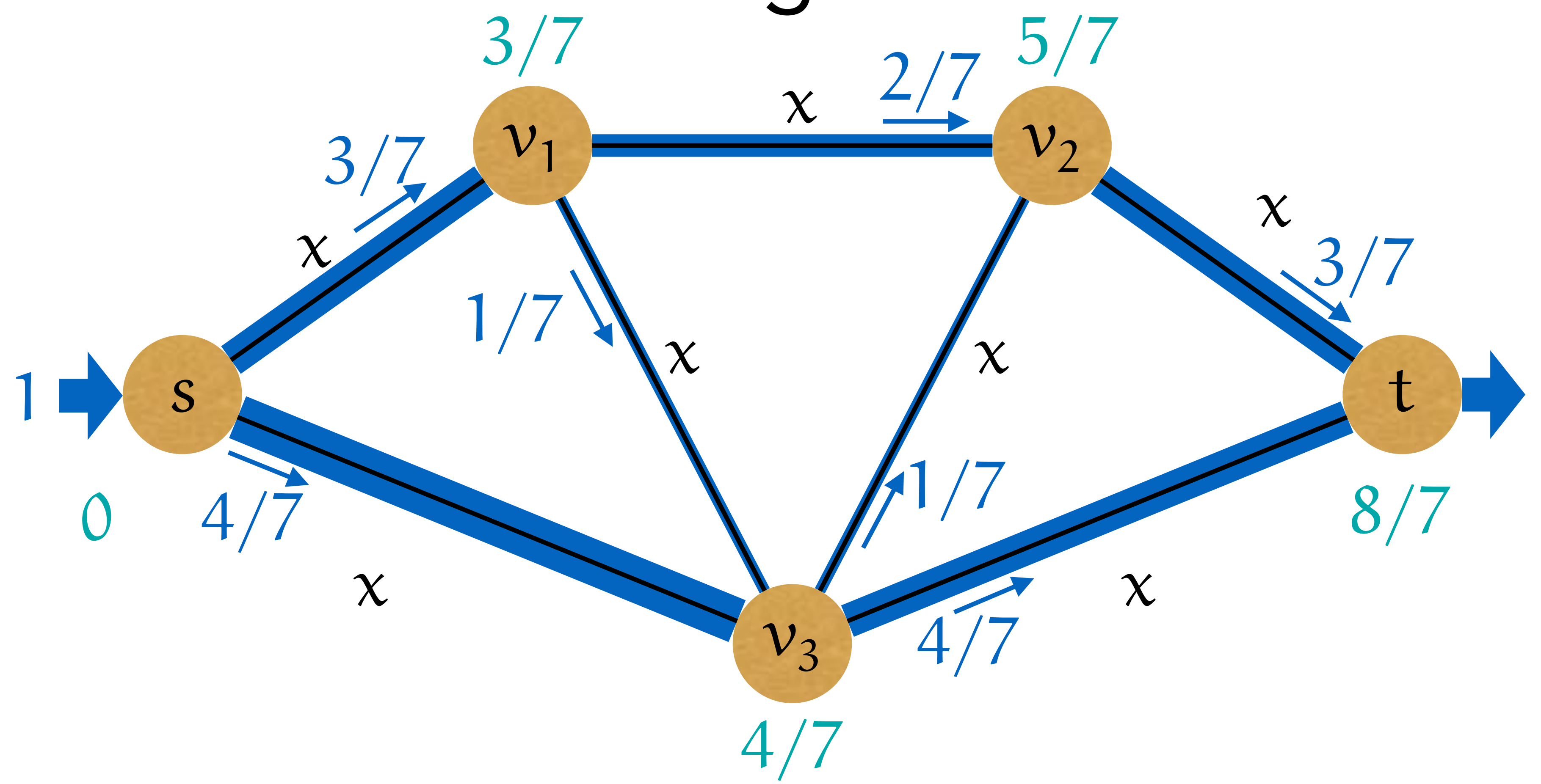

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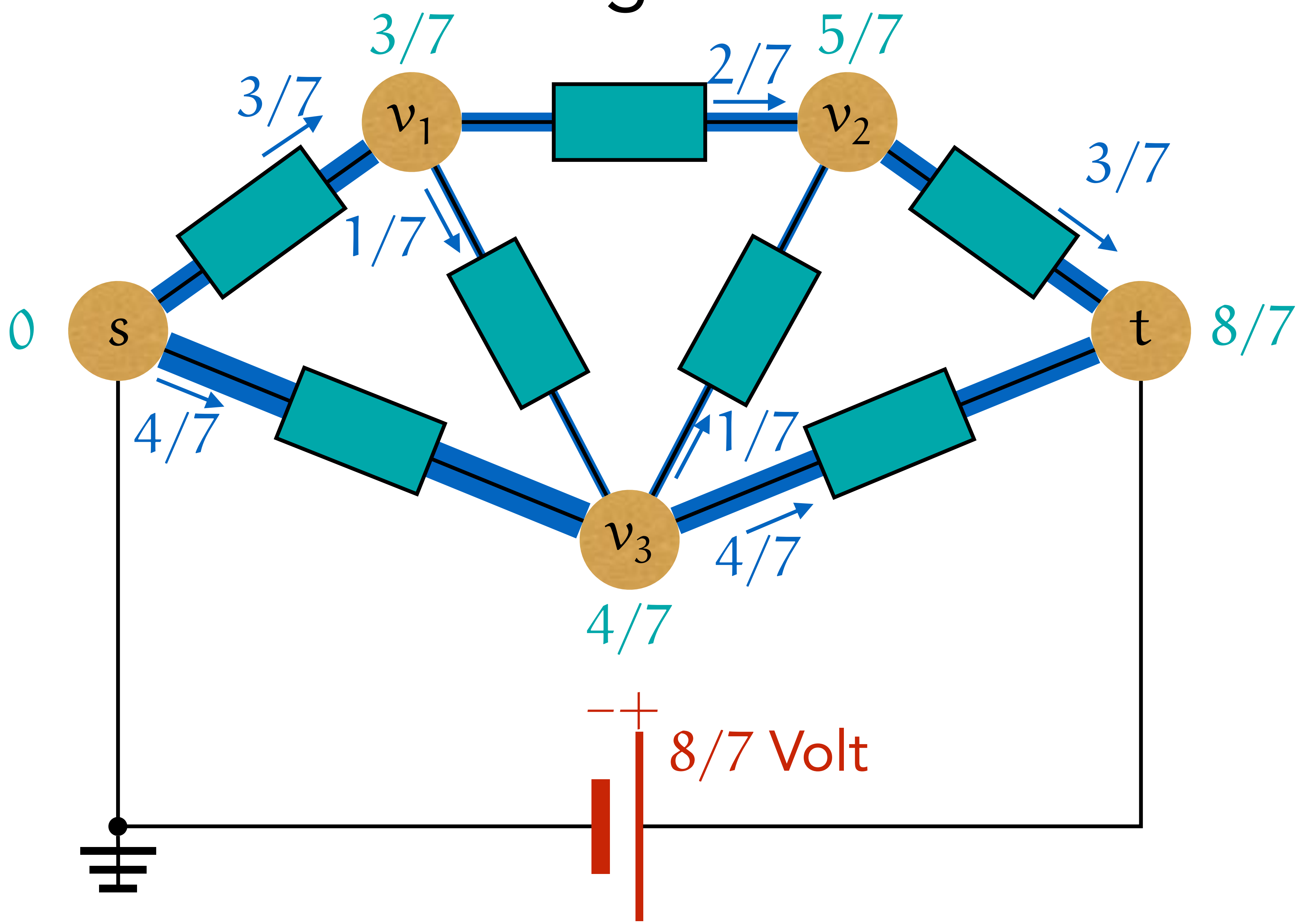
If $c_e(0) = 0$ for all e:

f WE $\Leftrightarrow \pi(w) - \pi(v) = c_e(f(v,w))$ for all edges.

Characterization of edge flows

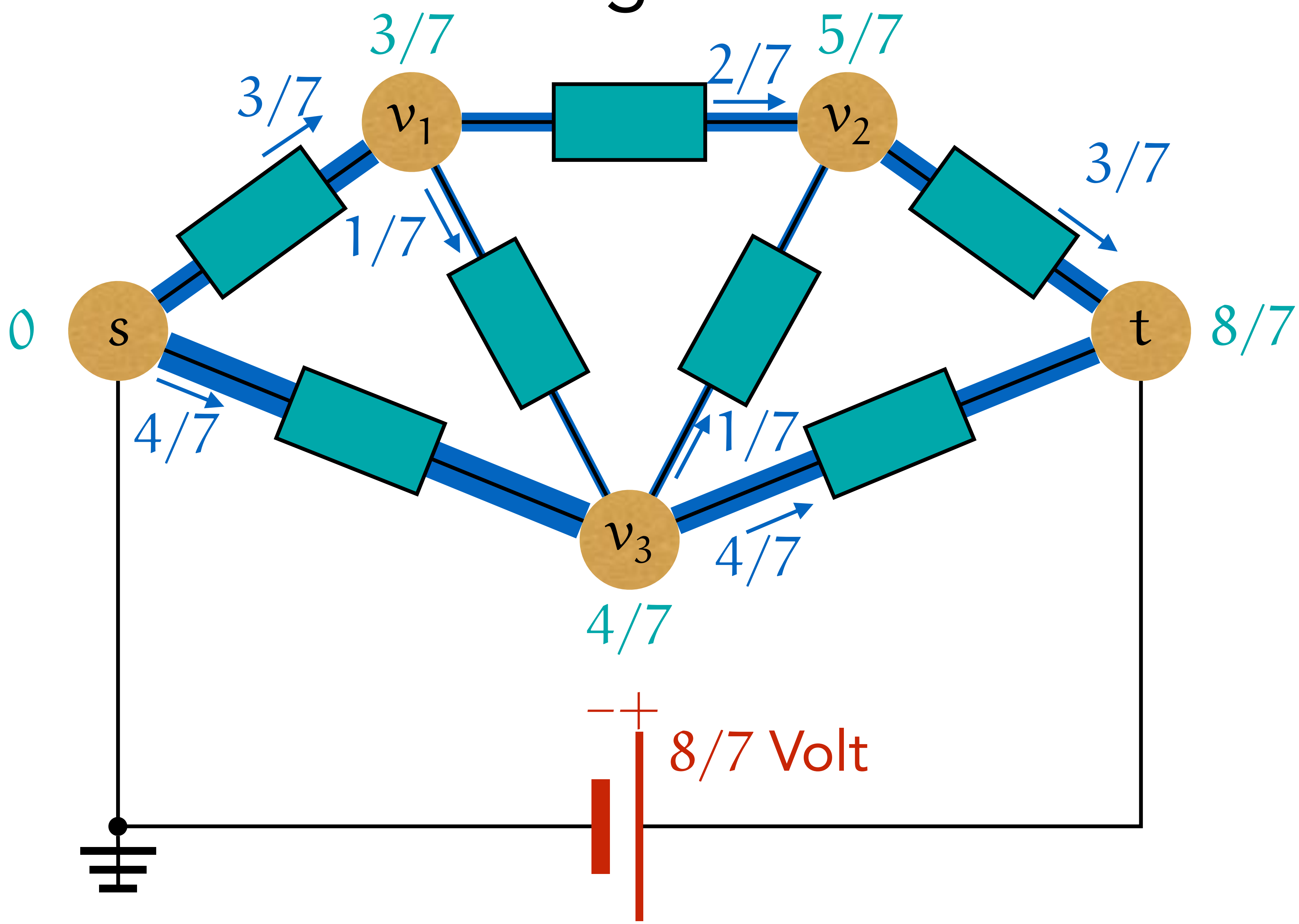


- Claim:

  the Wardrop equilibrium describes the electric current in a resistor network with a voltage of $8/7$.

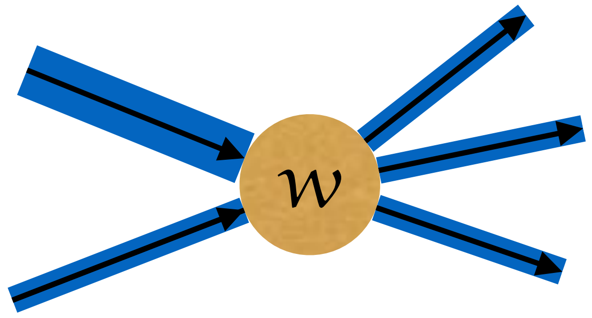
  (here: resistors with unit resistance)

Characterization of edge flows

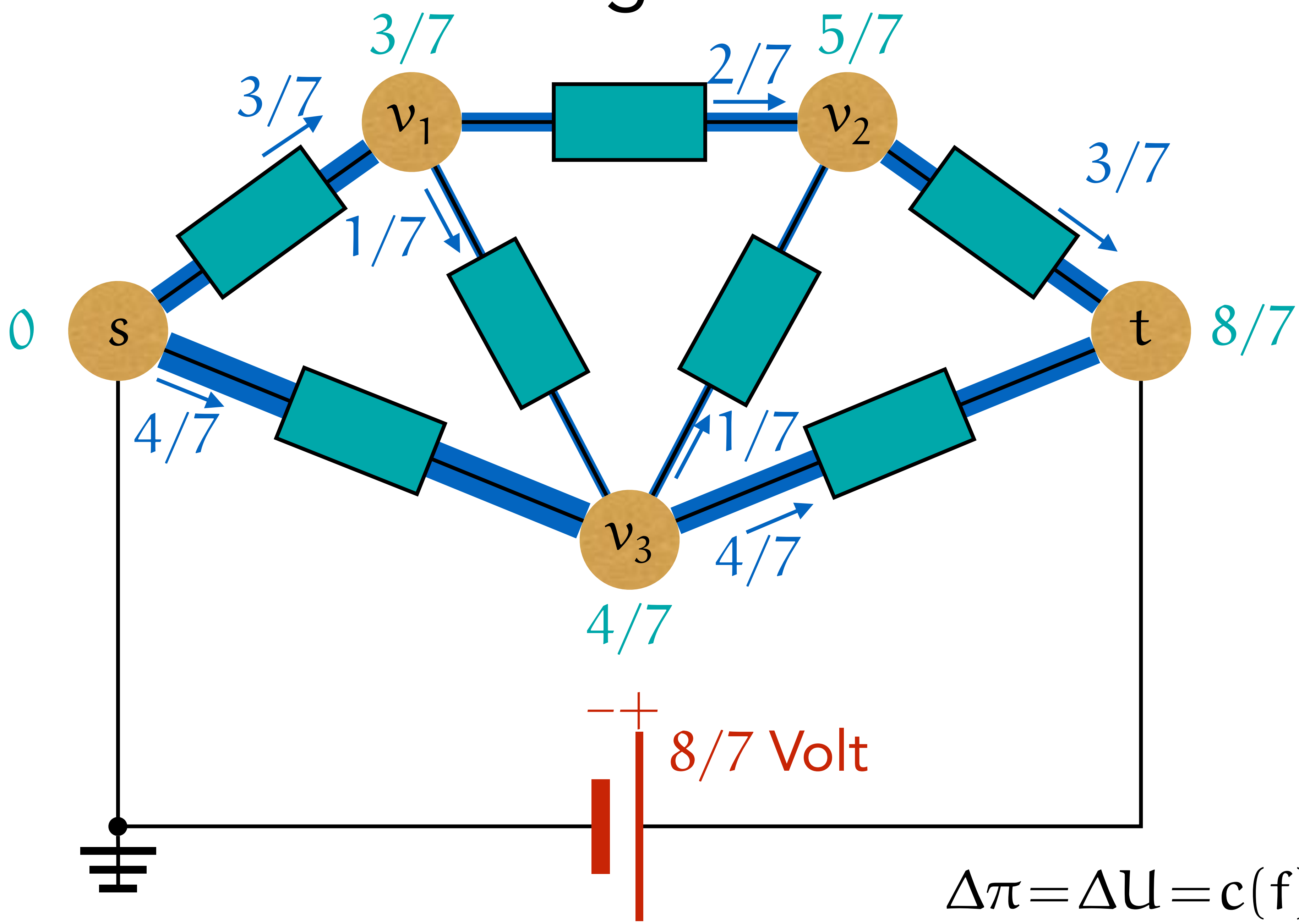


Kirchhoff's law ✔

At any node, inflow of current equals outflow of current.



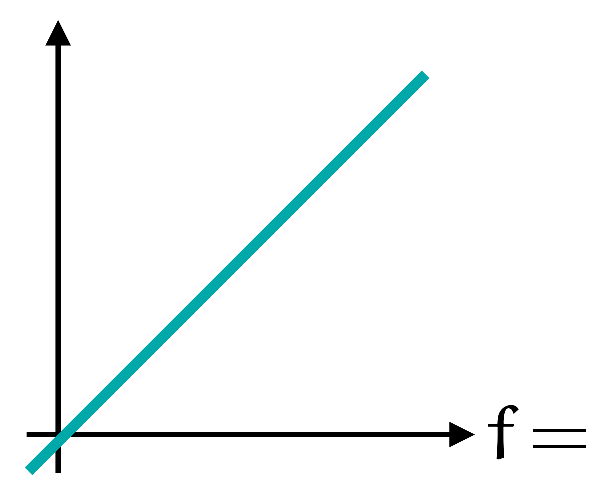
Characterization of edge flows



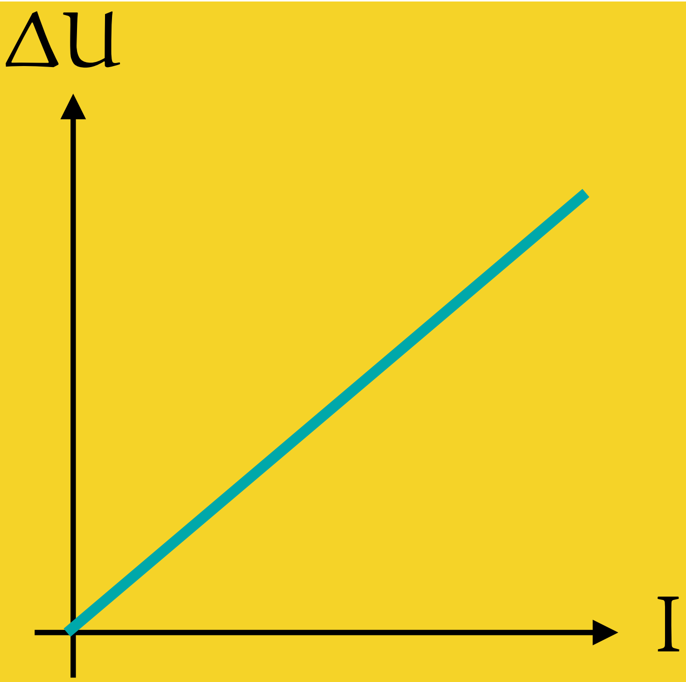
$\Delta\pi = \Delta U = c(f)$

Ohm's law ✔

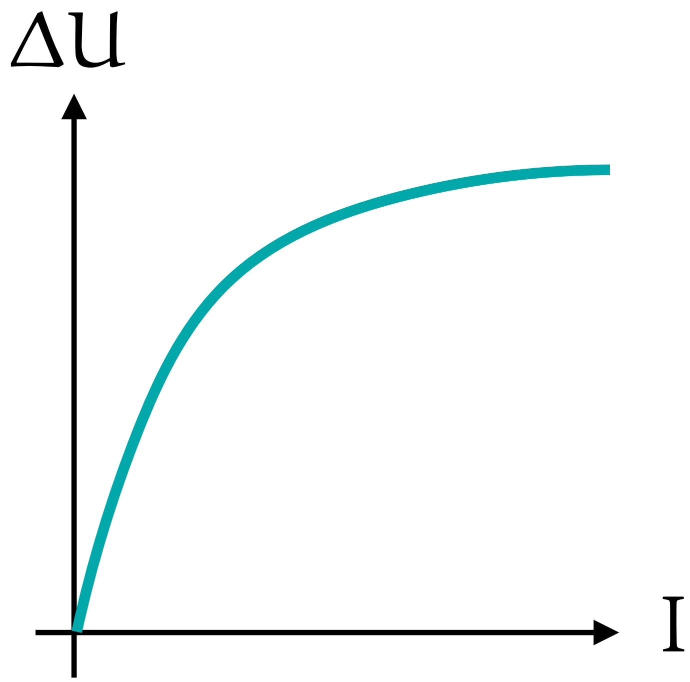
Current equals difference of voltage at end points over resistance.



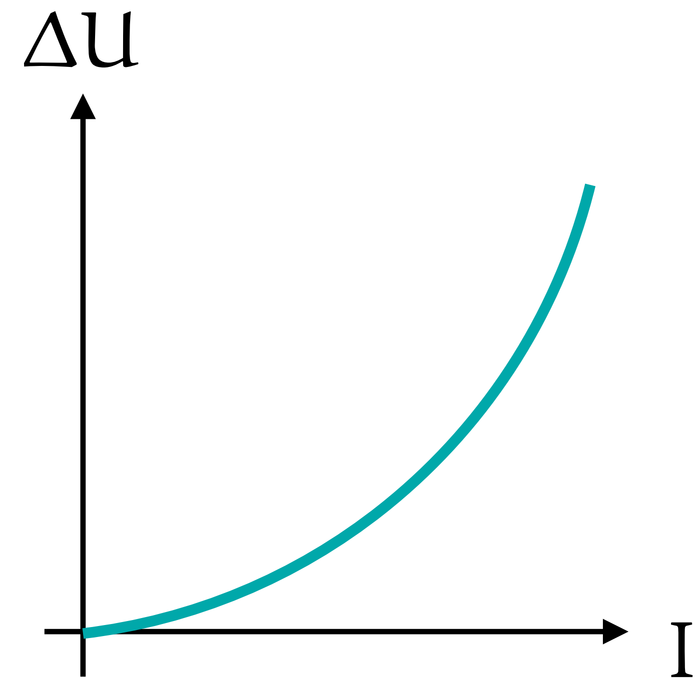
Notable characteristic curves



ohmic resistance

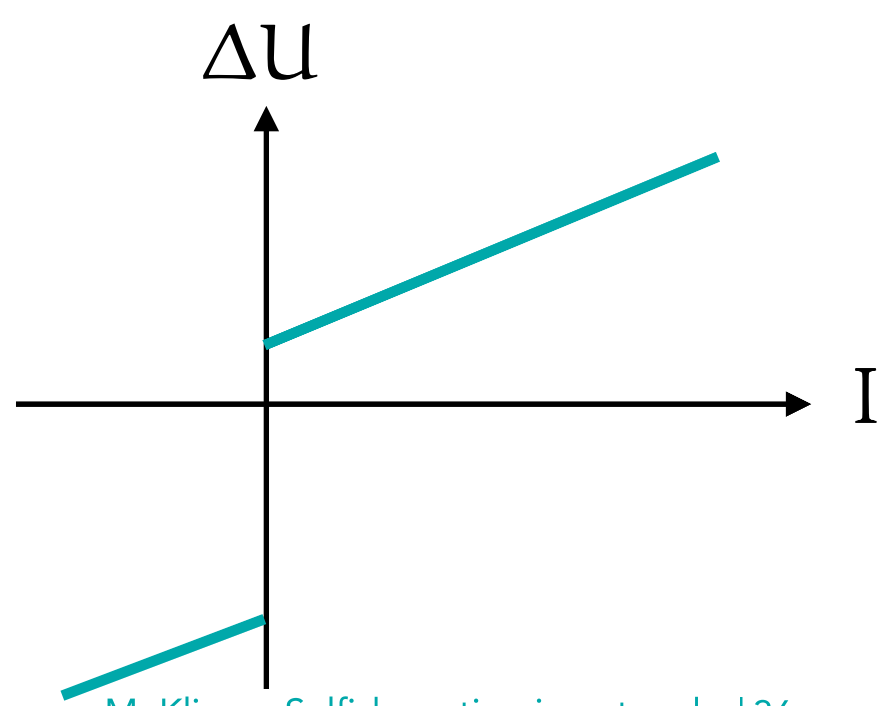


high-temperature conductor

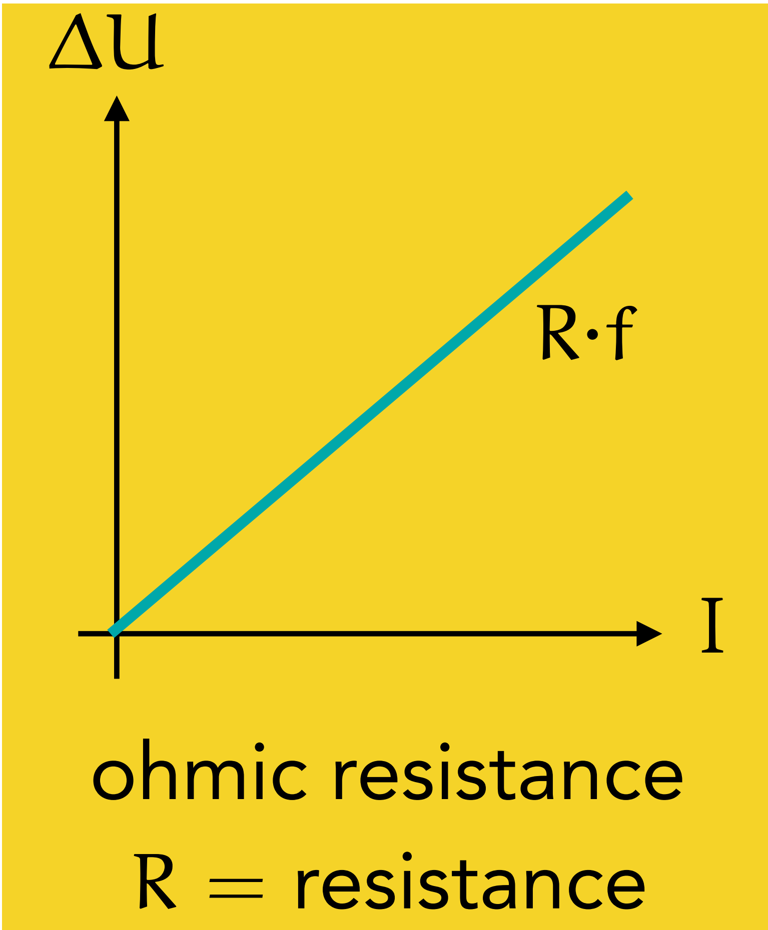


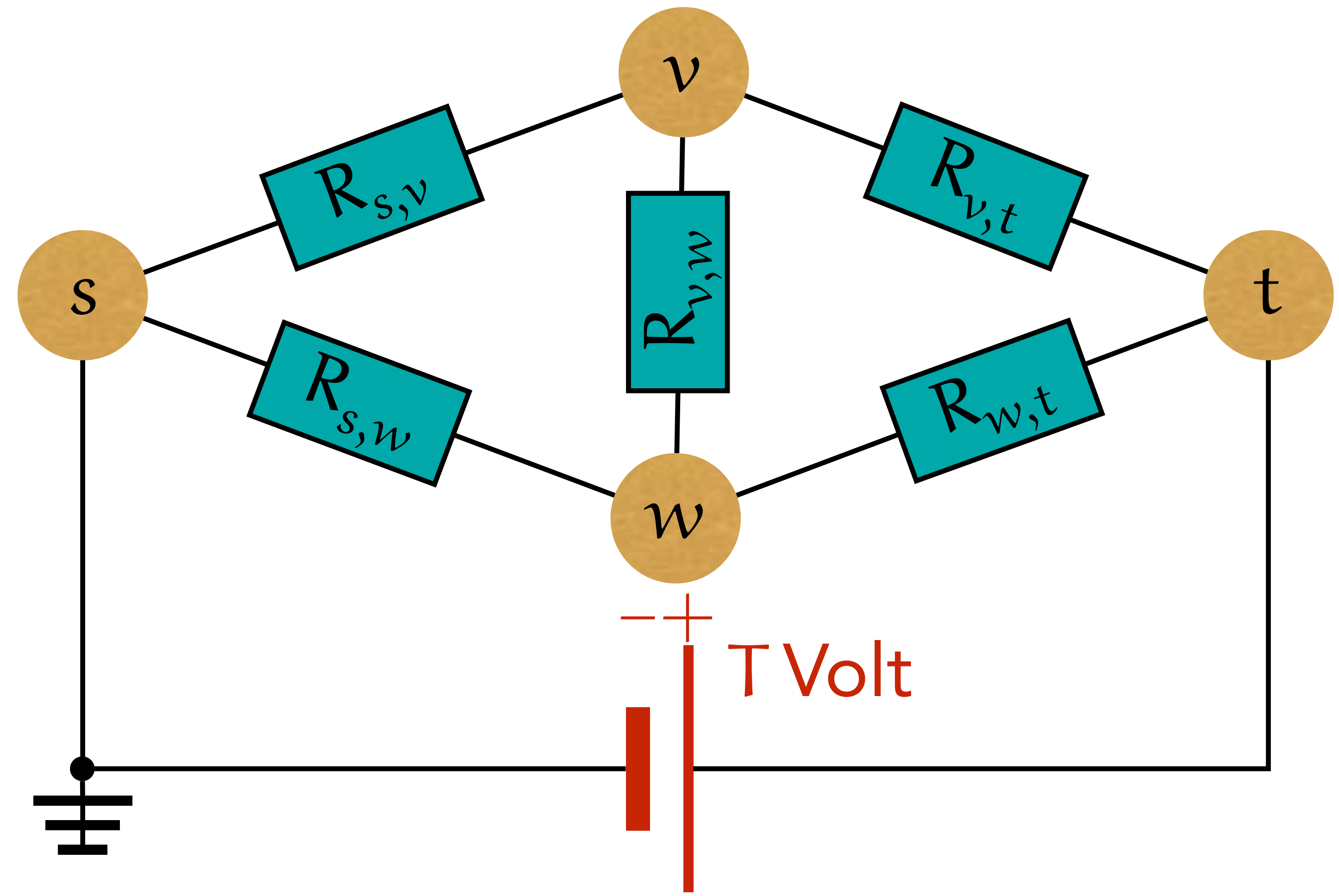
low-temperature conductor

- for these conductors, electric current is given by a WE
- there are other elements, e.g. diodes



Resistance networks





- Goal:

  easy computation of the electric current in the network

  (without computing a Wardrop equilibrium)

Easy computation of electric current

- (we allow here negative flows $f(v,w)$ corresponding to positive flows in the opposite direction)
- equilibrium condition: $f(v,w) = \alpha_{v,w}(\pi(w) - \pi(v))$

conductivity $\alpha_{v,w} = 1/R_{v,w}$

- flow conservation: $0 = \sum_{w \in \delta(v)} f(v,w)$

- $0 = \sum_{w \in \delta(v)} \alpha_{v,w}\big(\pi(w) - \pi(v)\big)$
- $\pi(v) \underbrace{\sum_{w \in \delta(v)} \alpha_{v,w}}_{A_v} = \sum_{w \in \delta(v)} \alpha_{v,w}\, \pi(w)$
- $\pi(v) \quad A_v \quad = \sum_{w \in \delta(v)} \alpha_{v,w}\, \pi(w)$

- $$\pi(v) = \sum_{w \in \delta(v)} \frac{\alpha_{v,w}}{A_v}\, \pi(w)$$
- Dirichlet problem with boundary conditions $\pi(s) = 0$ and $\pi(t) = T$.
- Fact: Solutions to Dirichlet problems are unique.

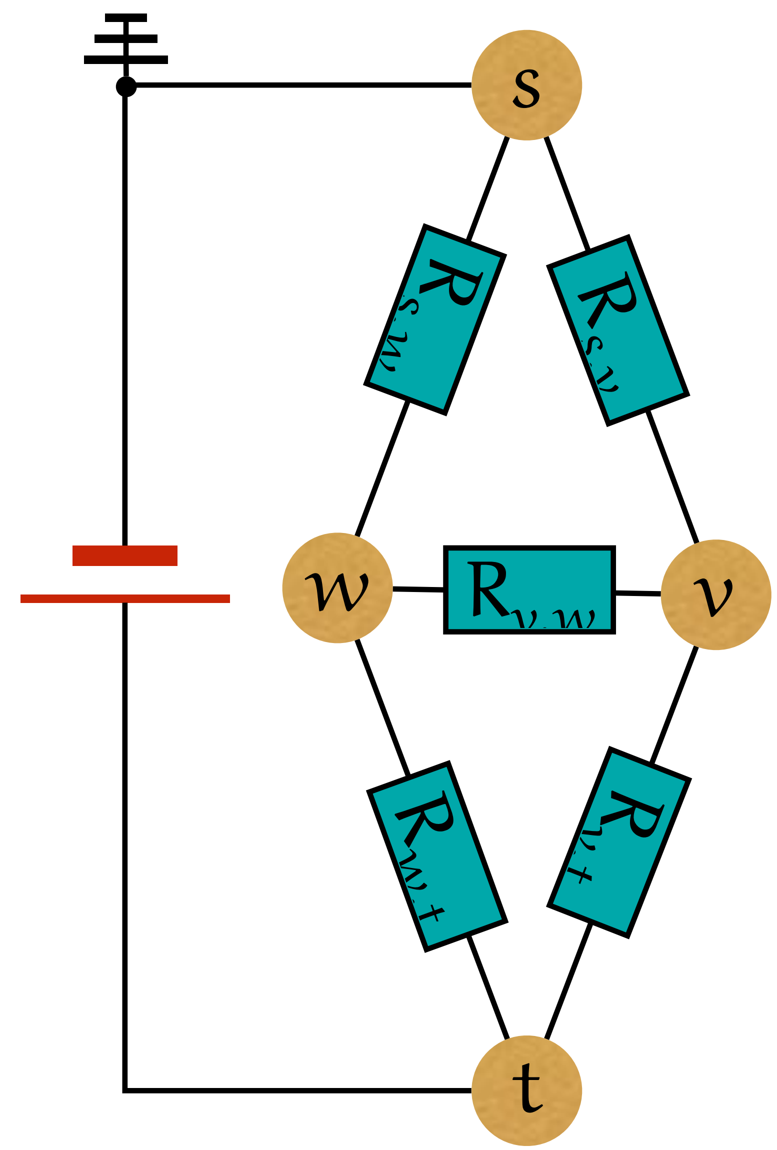
Interpretation as Markov chain

- Markov chain X on V
  with transition probabilities $\alpha_{v,w} / A_v$
- s and t are absorbing
  with payoffs $g(s)=0$ and $g(t)=T$
- $\varphi(v)=\mathbb{E}[g(u) \mid \text{stop in } u\in\{s,t\}, \text{start in } v]$

$$\pi(v) = \sum_{w\in\delta(v)} \frac{\alpha_{v,w}}{A_v} \pi(w)$$

$$\pi(s) = 0,\ \pi(t) = T$$

Dirichlet problem



Lemma

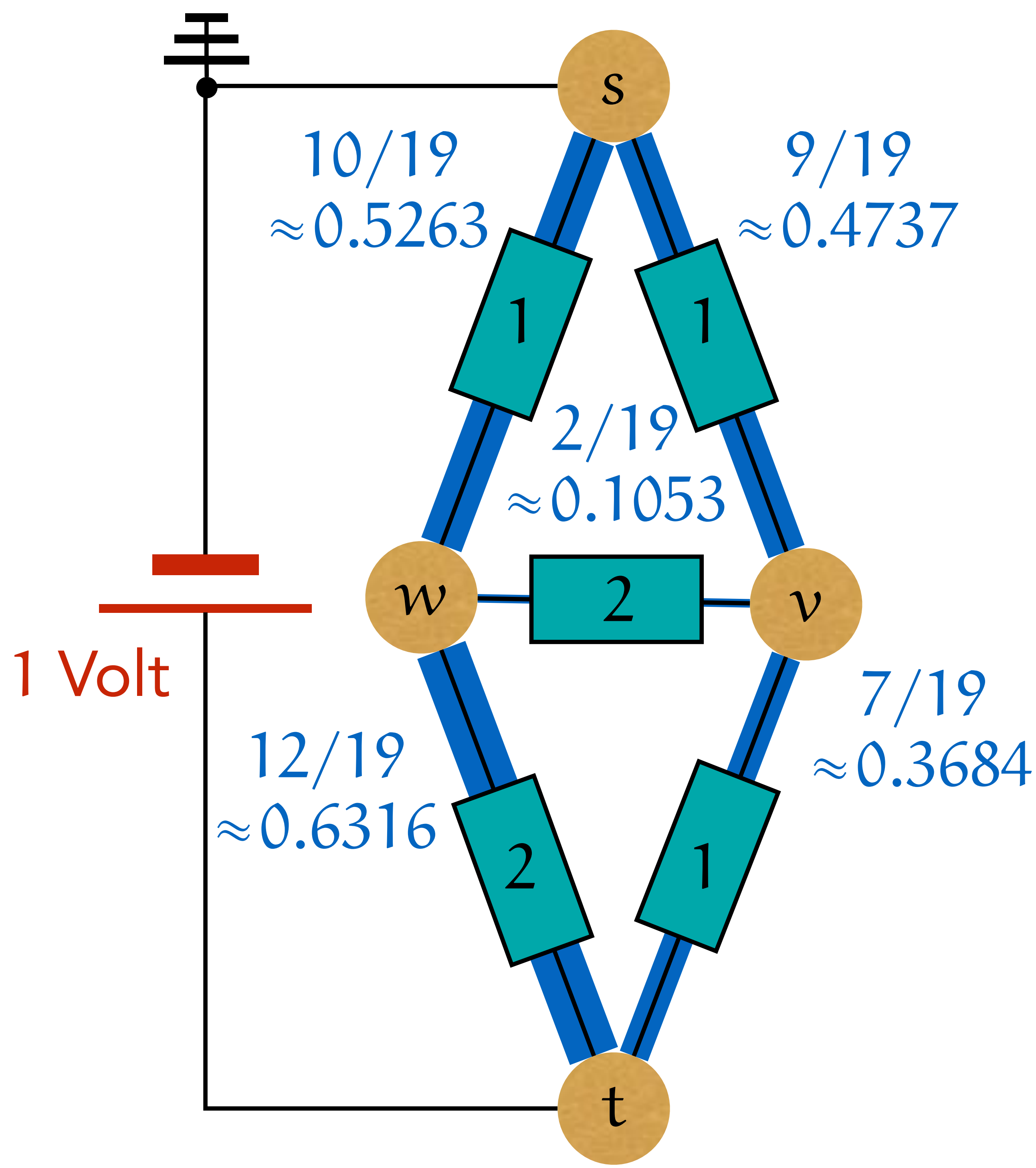
The expected payoffs $\varphi(v)$ are the unique solution of the Dirichlet problem.

Proof

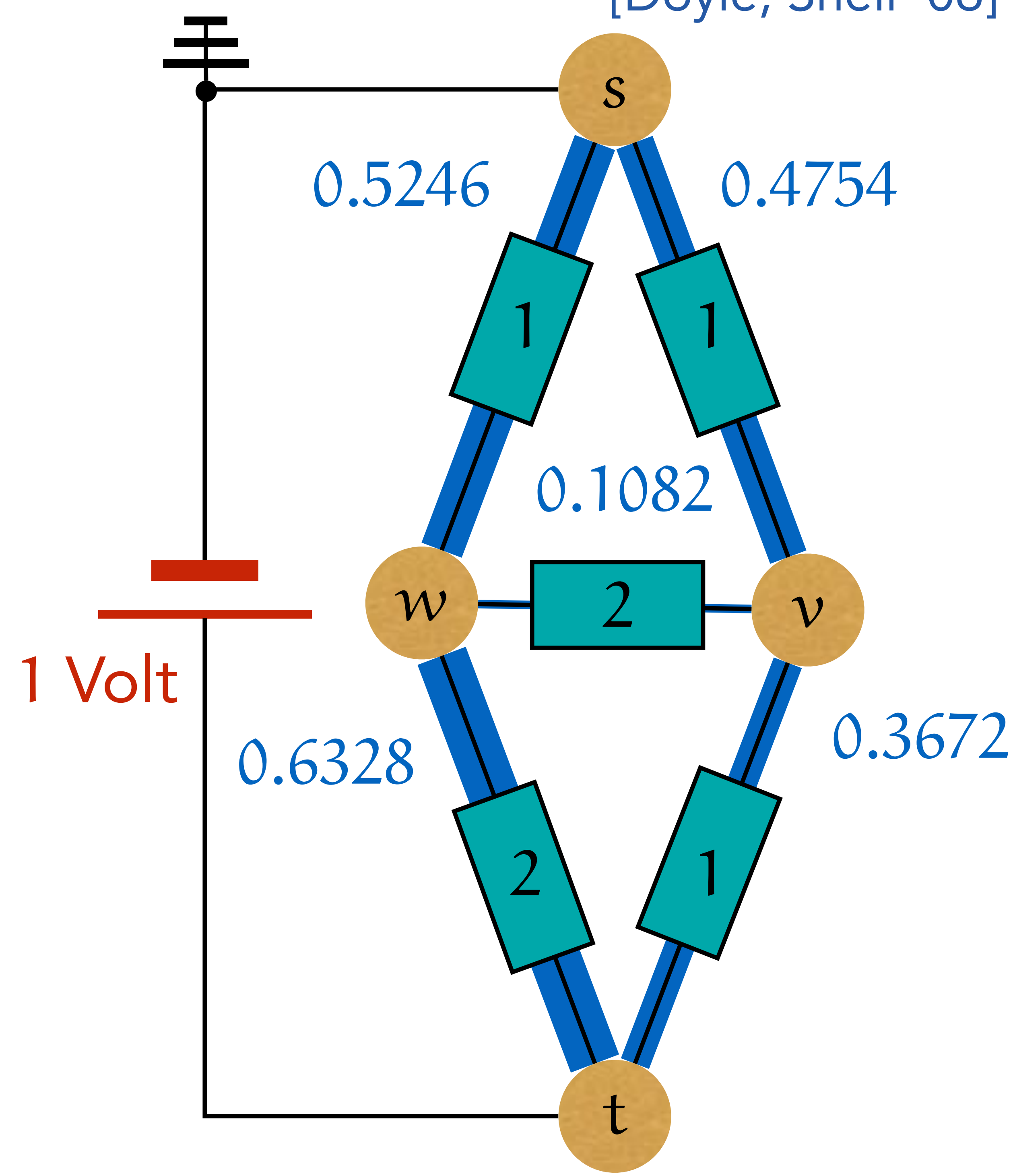
- $\varphi(v) = \mathbb{E}[g(u) \mid X_0=v]$
- $\varphi(v) = \sum_{w\in\delta(v)} \mathbb{E}[g(w) \mid X_0=v,\ X_1=w] \frac{\alpha_{v,w}}{A_v}$
- $\varphi(v) = \sum_{w\in\delta(v)} \varphi(w) \frac{\alpha_{v,w}}{A_v}$

Simulation

[Doyle, Snell ’06]



calculated currents



estimated currents
(10,000 random walks)

Consequences

- Thompson's Principle: [Thompson, Tait, 1879]

  Electric flow minimizes energy dissipation $1/2 \sum_{e \in E} R_e \, f(e)^2$

  - Proof: Electric flow is WE with cost functions $c_e(x) = R_e \, x$, thus minimizes $\sum_{e \in E} \int_0^{g_e} c_e(t) \, dt = 1/2 \sum_{e \in E} R_e \, g(e)^2$.

- Effective resistance:

  A network behaves like a single resistor with resistance R_{eff}.

  - Proof: Flows and potentials are scale-invariant .

- Rayleigh's Monotonicity Law:

  Increasing single resistances cannot decrease effective resistance.

  - Proof: $R_{eff} = 1/2 \sum_{e \in E} R_e \, f(e)^2$, and the latter cannot be decreased when increasing resistances.

- Rayleigh's Monotonicity Law, in turn, implies similar statements for random walks.

Equilibrium flows

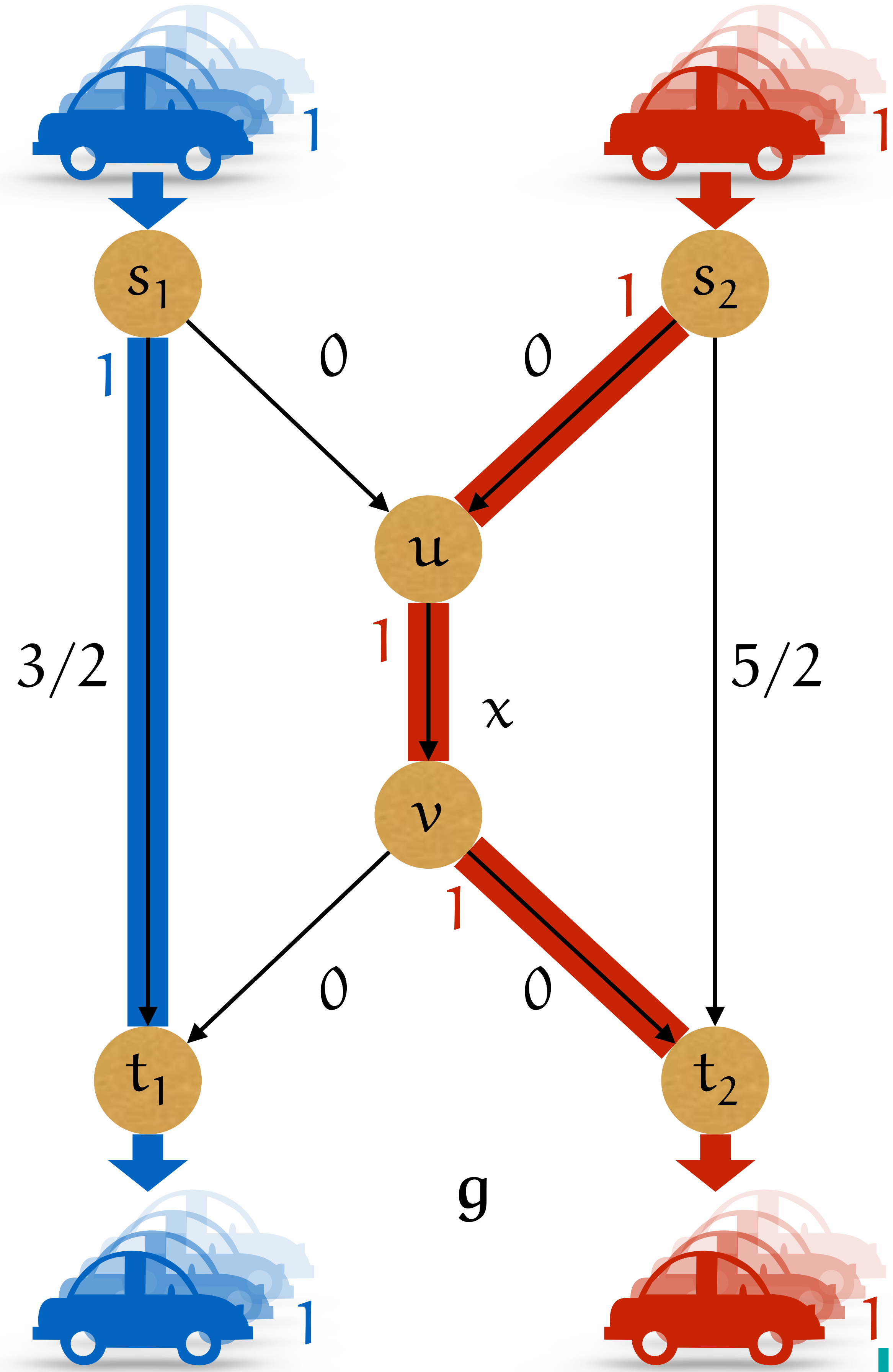
Relationship with system optimum

System-optimal flows

- total travel time

  $$C(f) = \sum_{i \in K} \sum_{P \in \mathcal{P}i} f_i(P) \sum_{e \in P} c_e(f(e)) = \sum_{e \in P} c_e(f(e)) f(e)$$

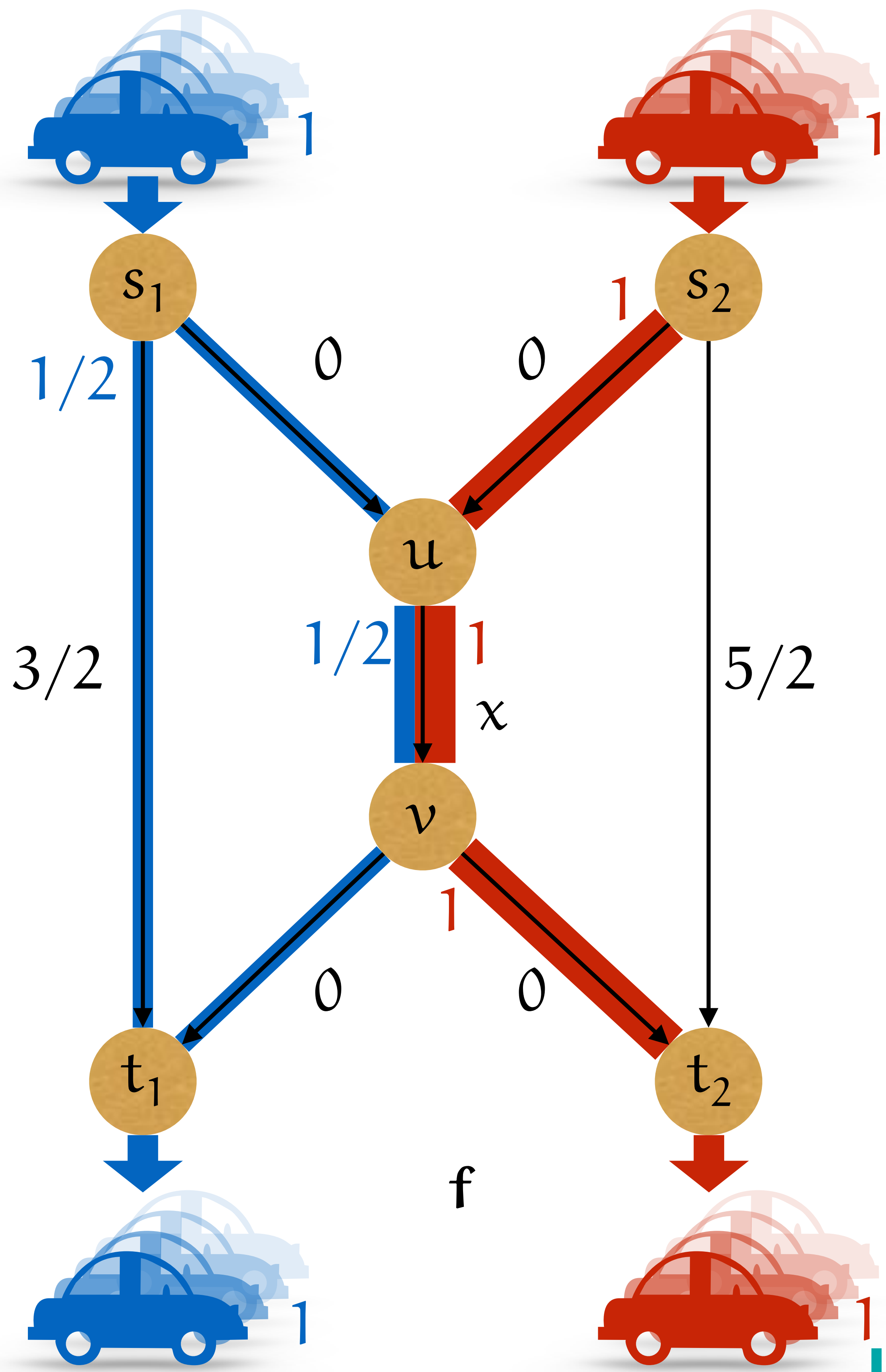
- $C(g) = 3/2 + 1 = 5/2$



System-optimal flows

- total travel time

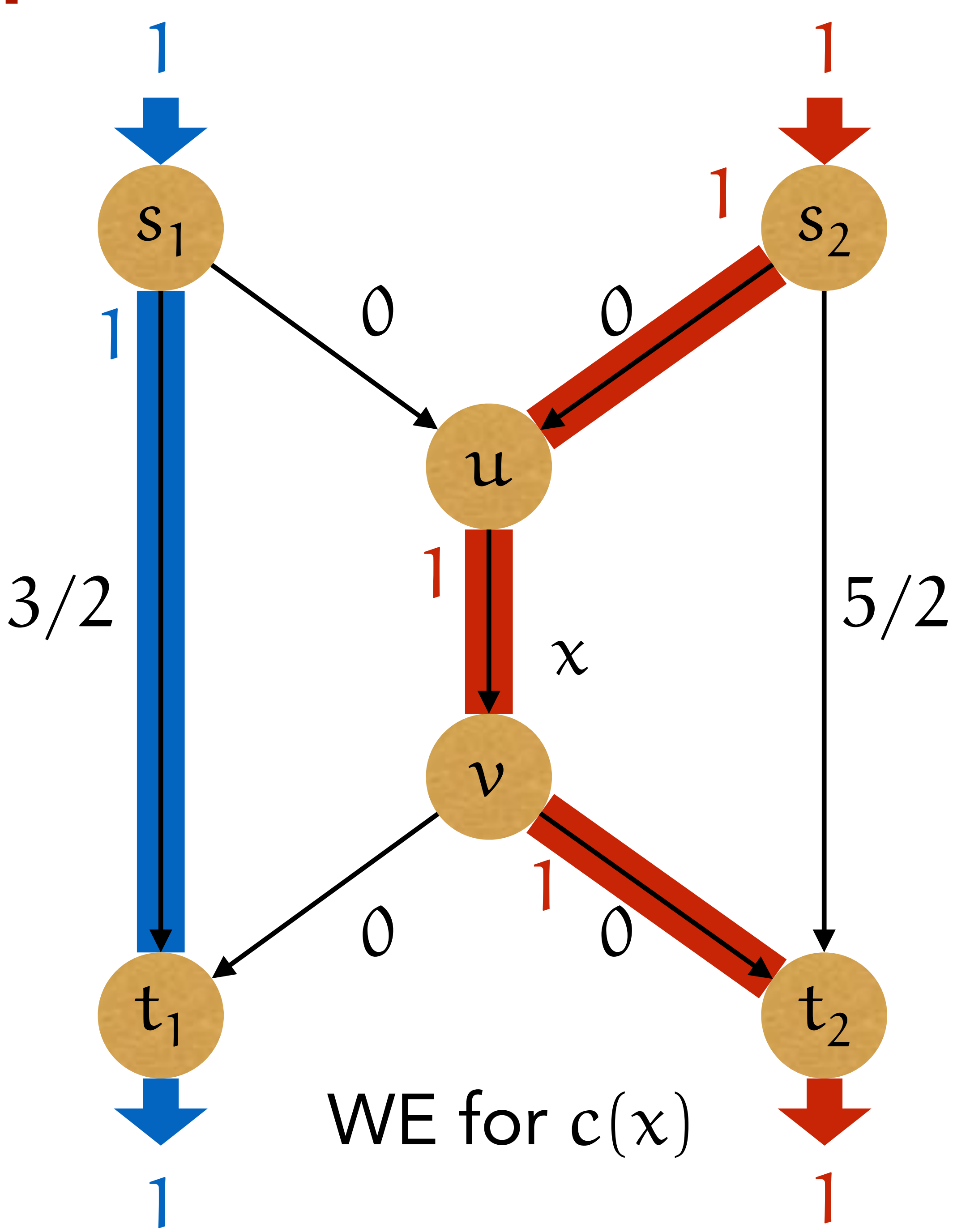
  $C(\mathbf{f}) = \sum_{i \in K} \sum_{P \in \mathcal{P}_i} f_i(P) \sum_{e \in P} c_e(f(e))$
  $= \sum_{e \in E} c_e(f(e)) f(e)$
- $C(\mathbf{g}) = 3/2 + 1 = 5/2$
- $C(\mathbf{f}) = 3/4 + 9/4 = 3$
- Wardrop equilibrium need not minimize the total travel time

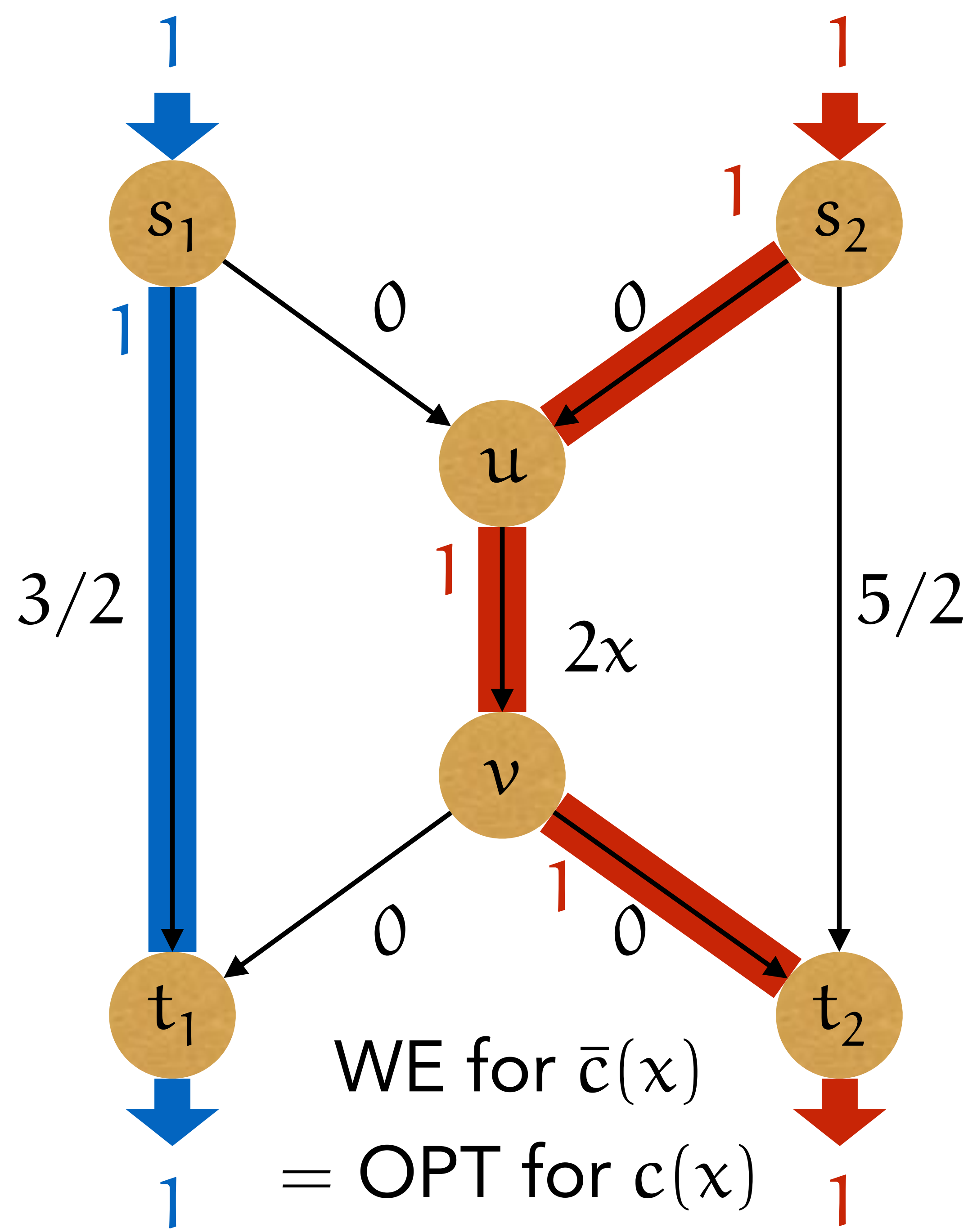


Characterization of system-optimal flows

Theorem [Beckman et al. '56]

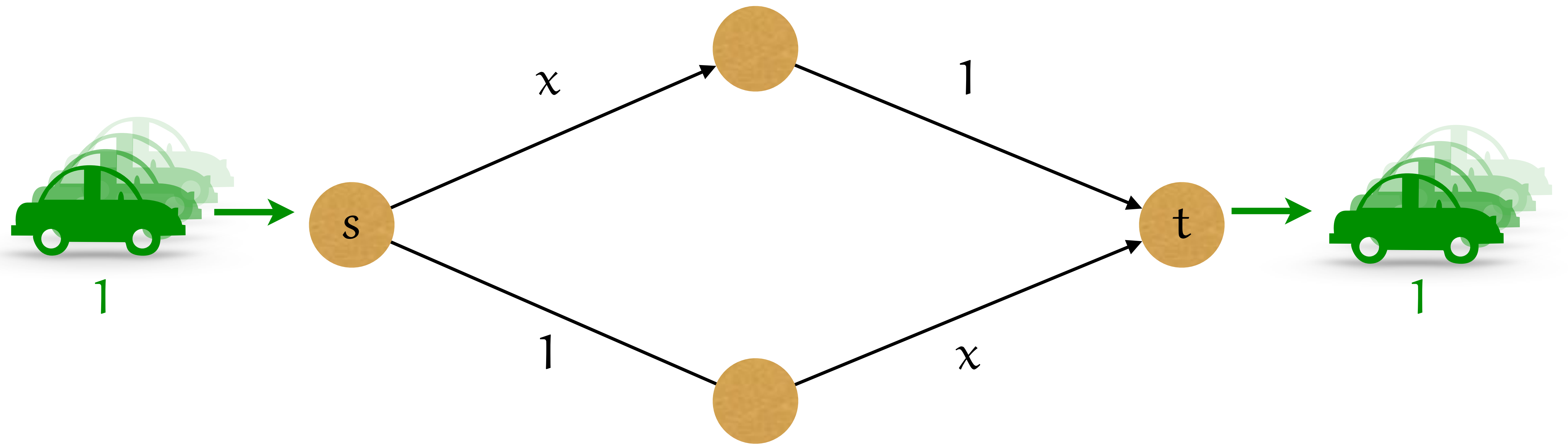
Flow f is system-optimal if and only if it is a Wardrop equilibrium for the modified cost functions $\bar{c}(x) = c(x) + c'(x)x$.





Braess' Paradox

[Braess '68]



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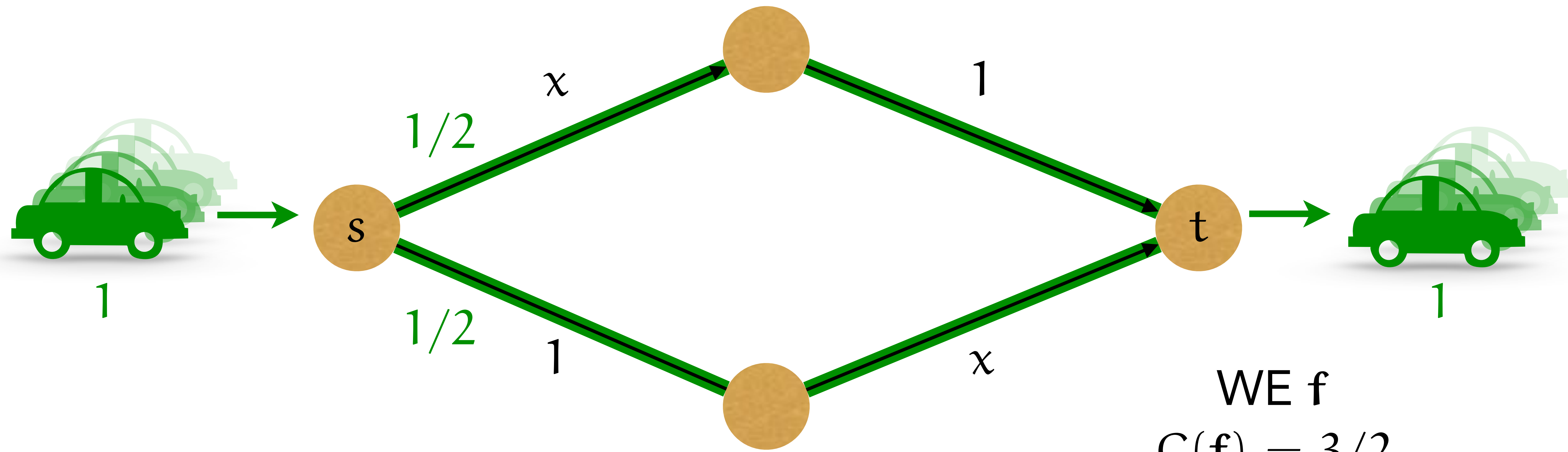
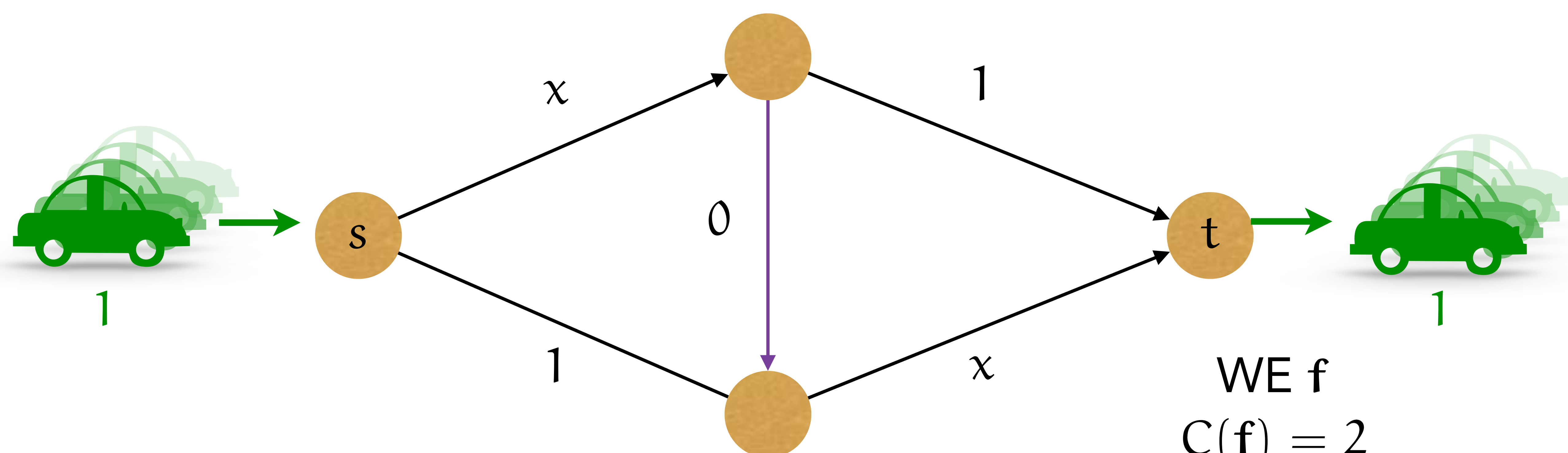


WE f

$C(\mathbf{f}) = 3/2$

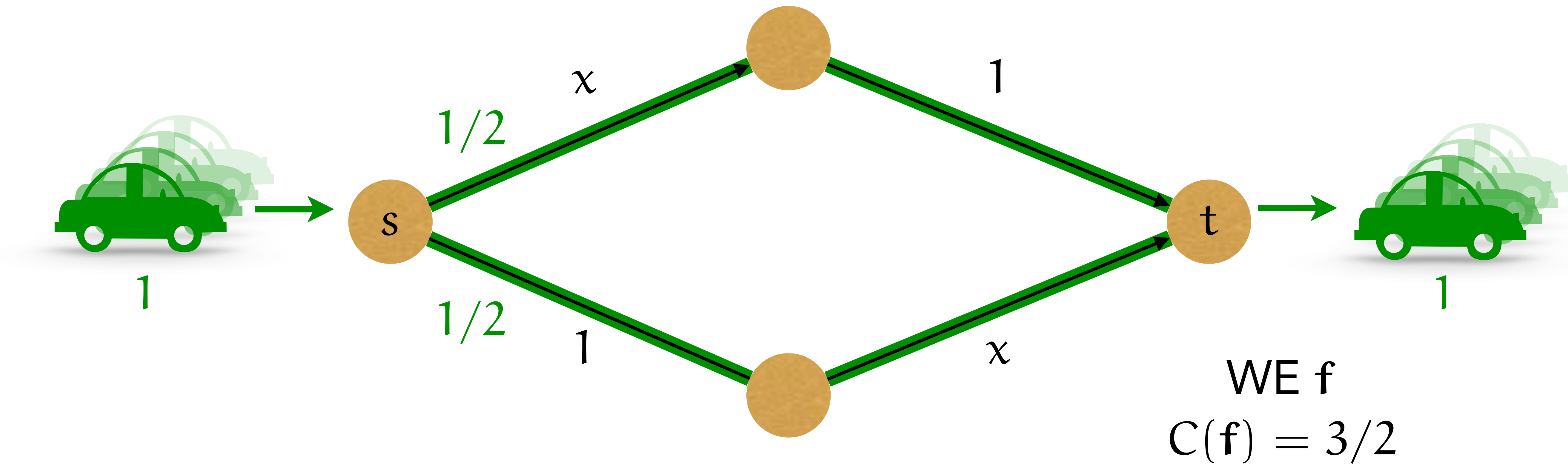


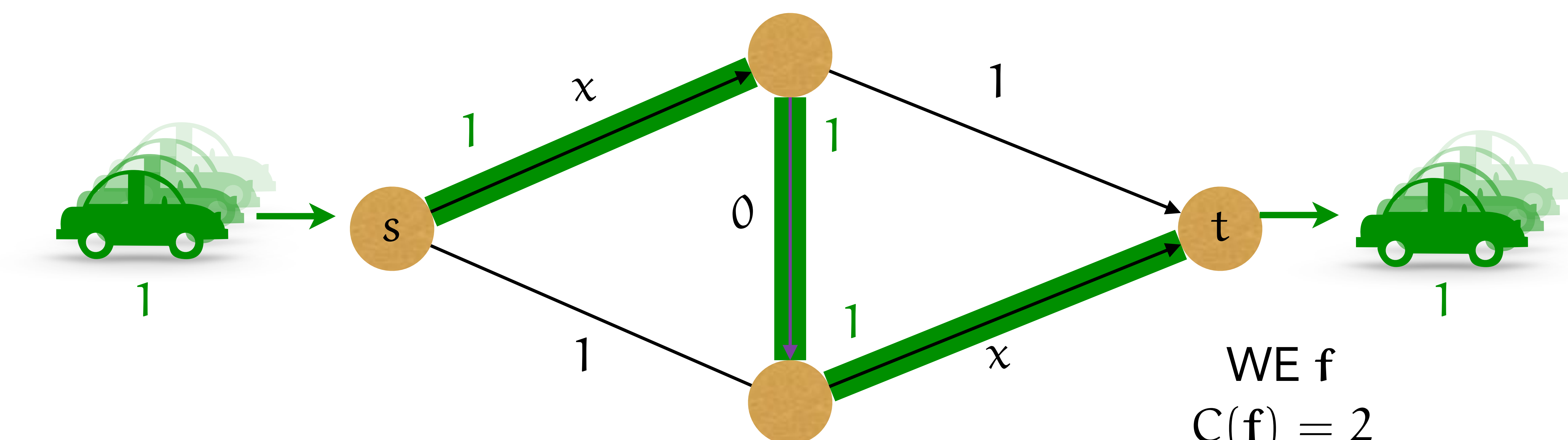
WE f

$C(\mathbf{f}) = 2$

Braess' Paradox

[Braess '68]





- no Rayleigh Law for transportation networks!

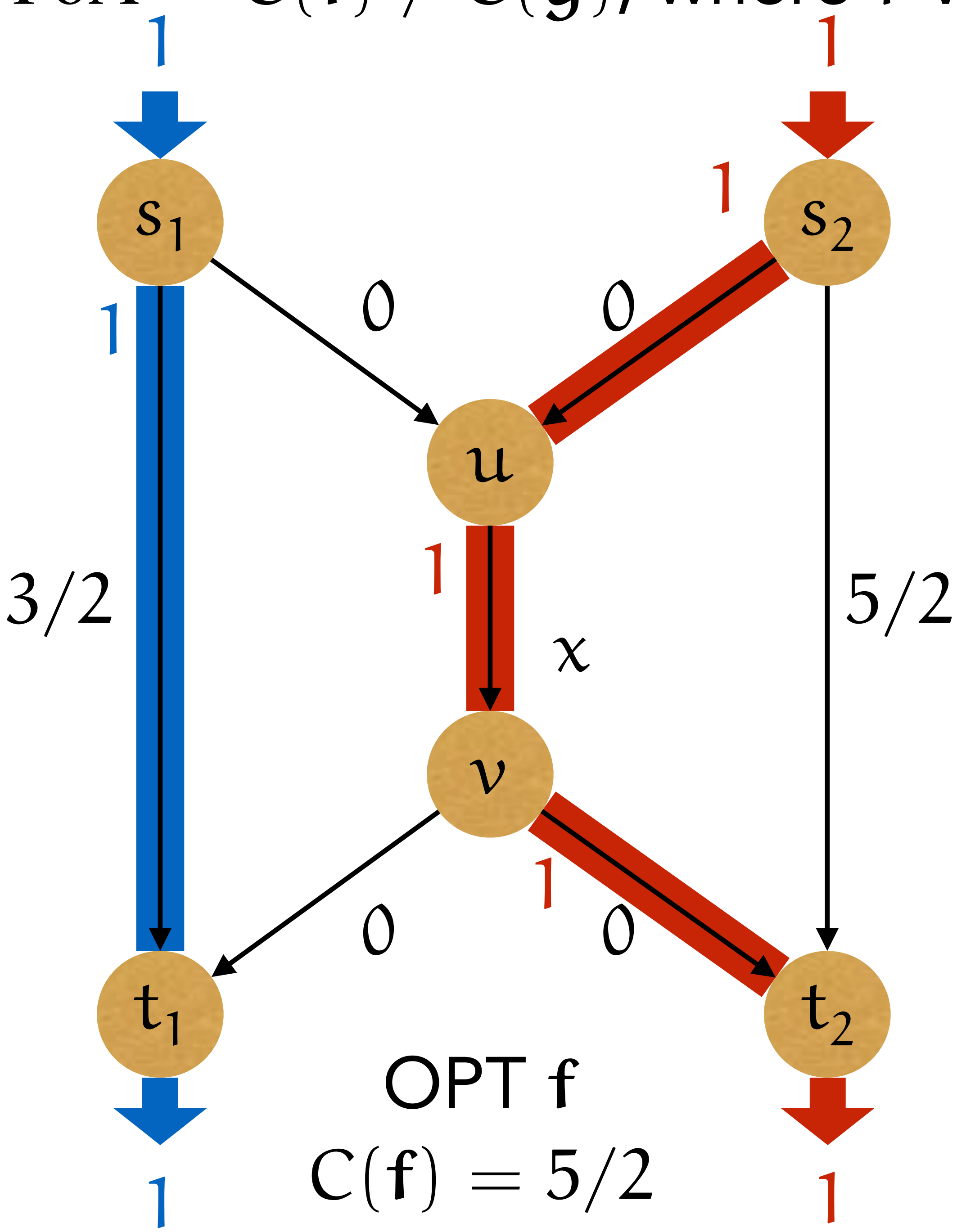
Equilibrium flows

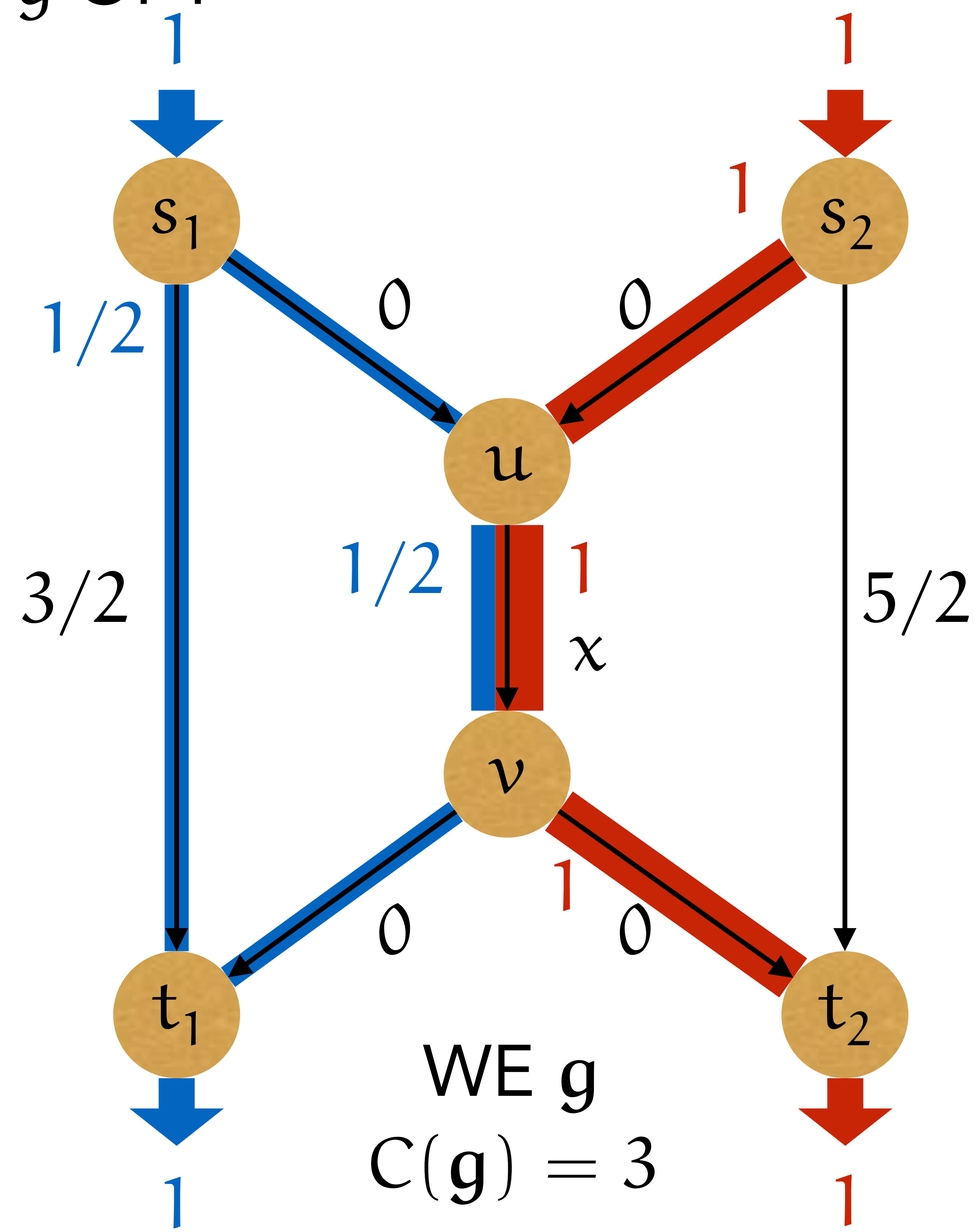
Efficiency

Price of anarchy

[Koutsoupias, Papadimitriou '99; Papadimitriou '01]

- price of anarchy measures the efficiency due to lack of coordination
- $\text{PoA} = C(\mathbf{f}) \,/\, C(\mathbf{g})$, where $\mathbf{f}$ WE and $\mathbf{g}$ OPT





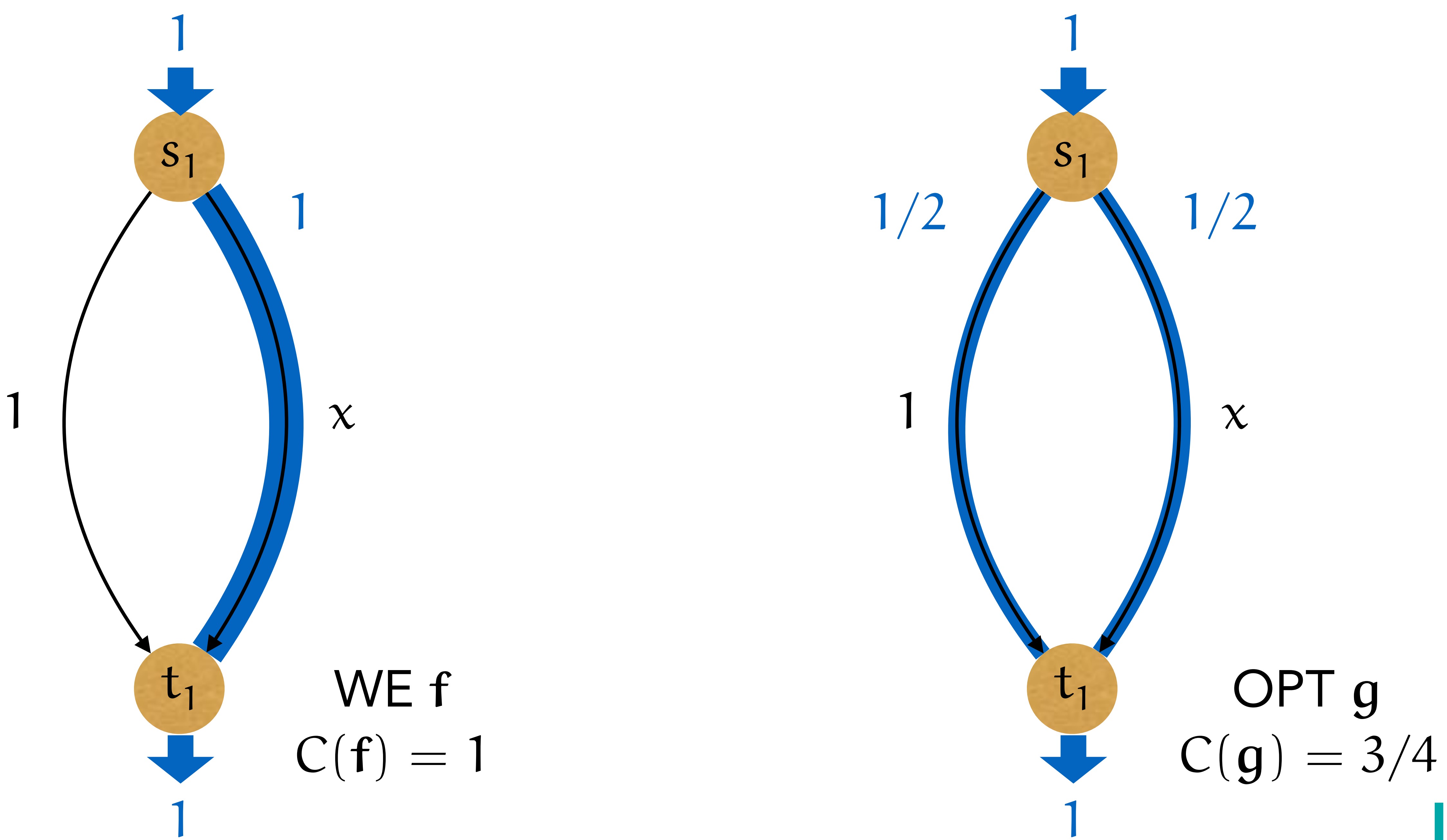
- $\text{PoA} = 6/5 = 1.2$

Price of anarchy of affine costs

Theorem [Roughgarden, Tardos '02]

$\mathrm{PoA} \leqslant 4/3$ for all networks with affine costs $c(x) = ax + b$; $a, b \in \mathbb{R}_+$.

- this bound is tight [Pigou 1920]



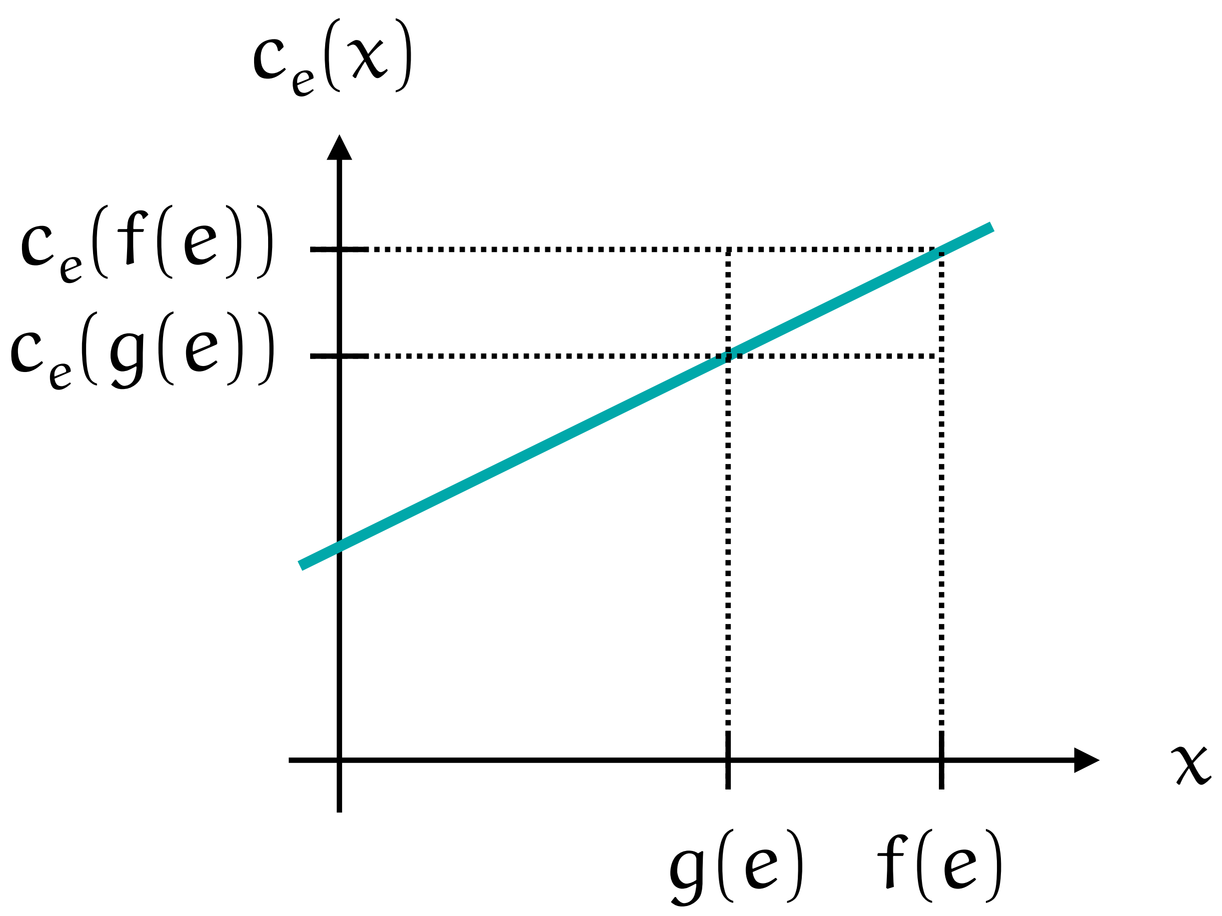
Proof of the upper bound

[Correa et al. '08]

- $C(\mathbf{f}) = \sum_{e \in E} c_e(f(e))\, f(e)$

  $\leqslant \sum_{e \in E} c_e(f(e))\, g(e)$ (for OPT $\mathbf{g}$, by VI)

  $\leqslant \sum_{e \in E} c_e(g(e))\, g(e) + \sum_{e \in E} \big(c_e(f(e)) - c_e(g(e))\big)\, g(e)$



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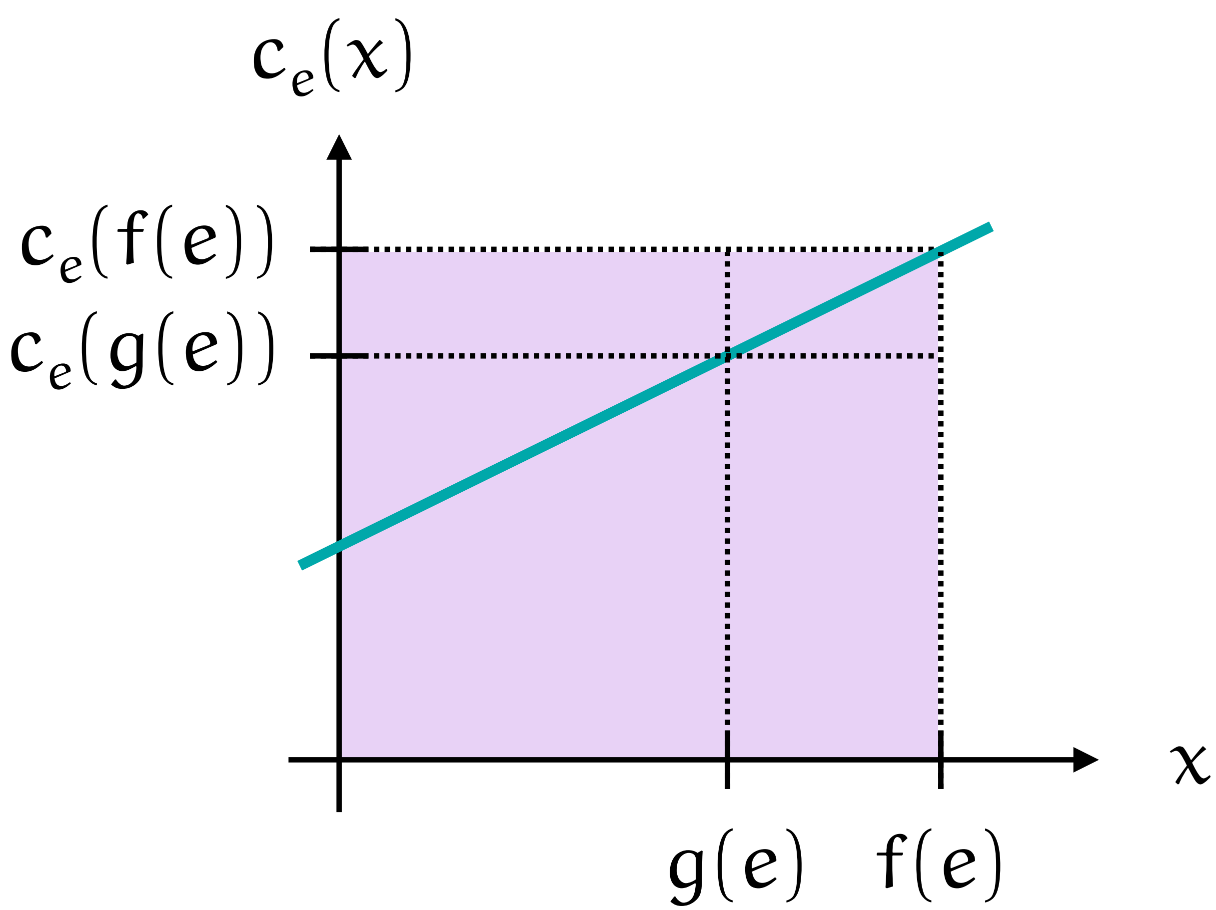

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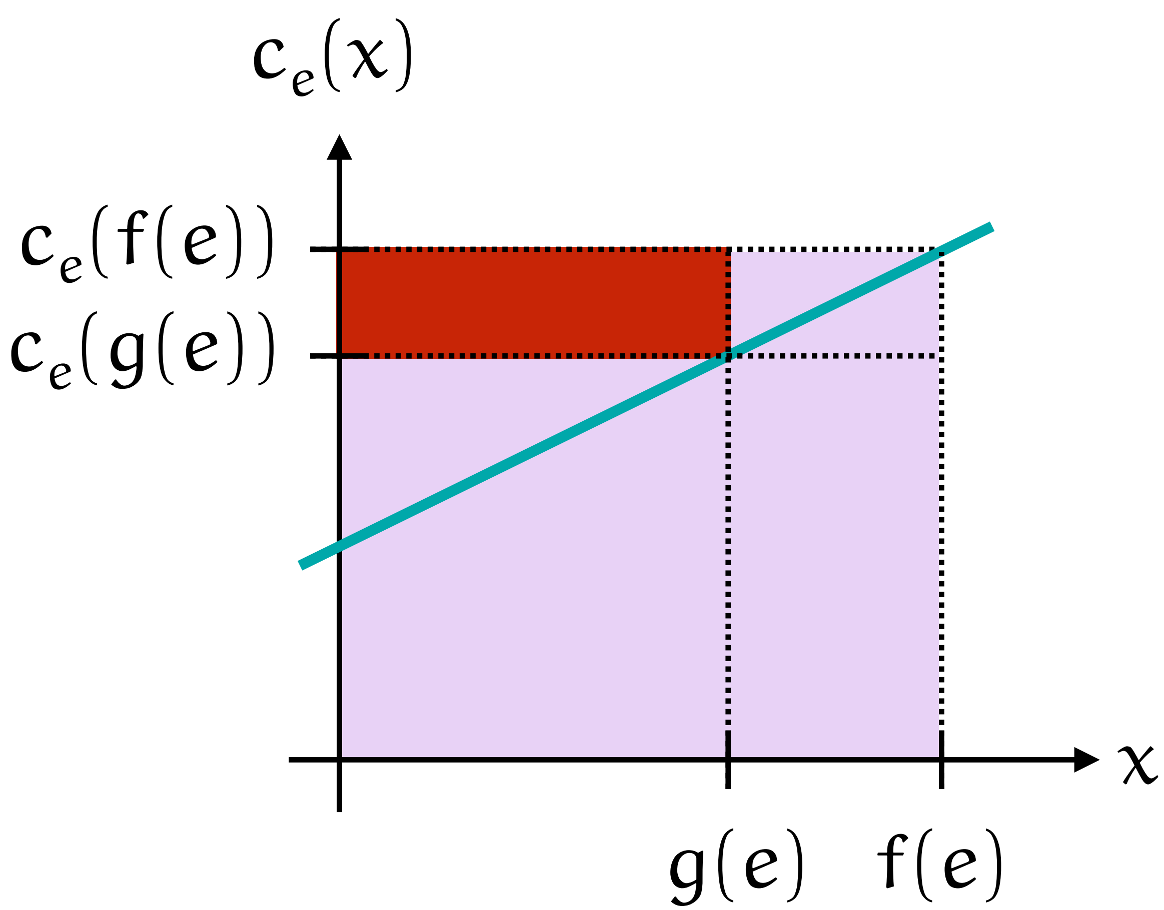
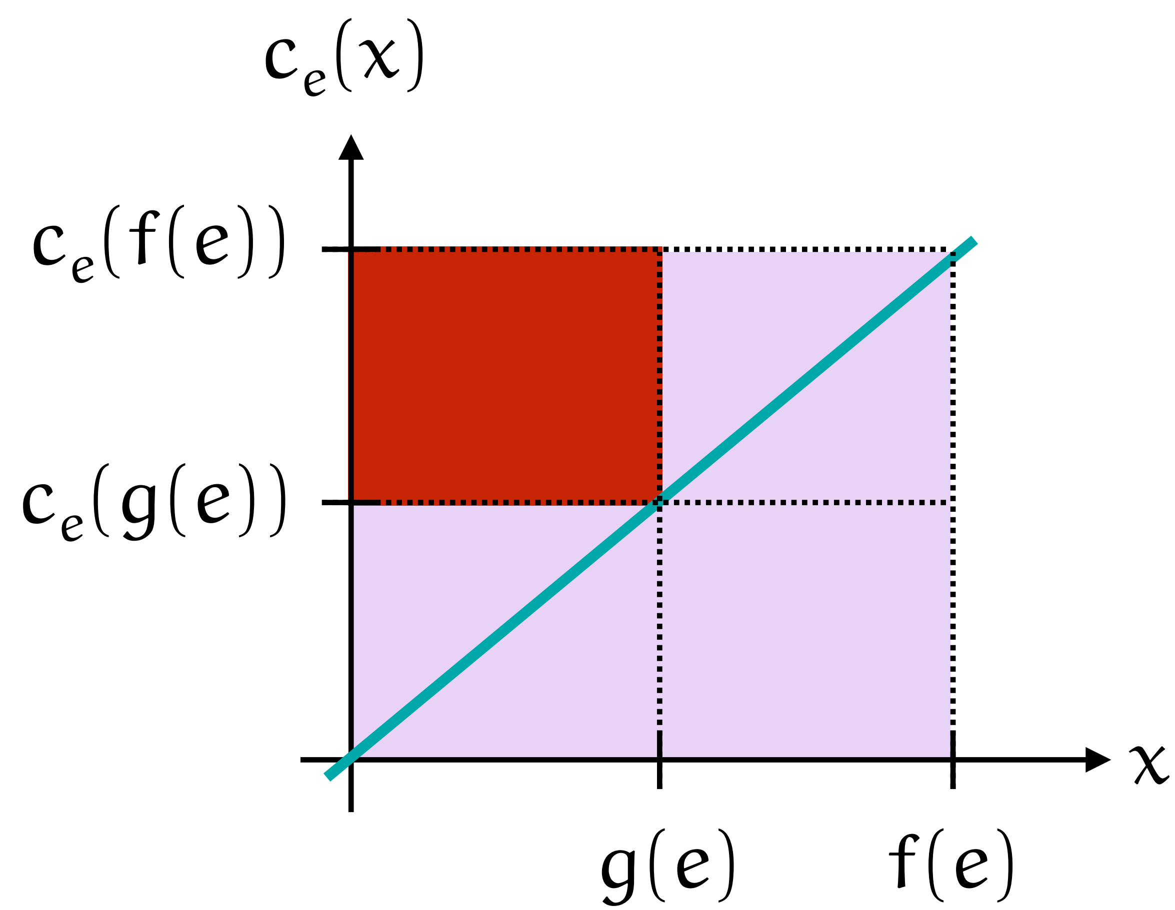


$\leqslant$



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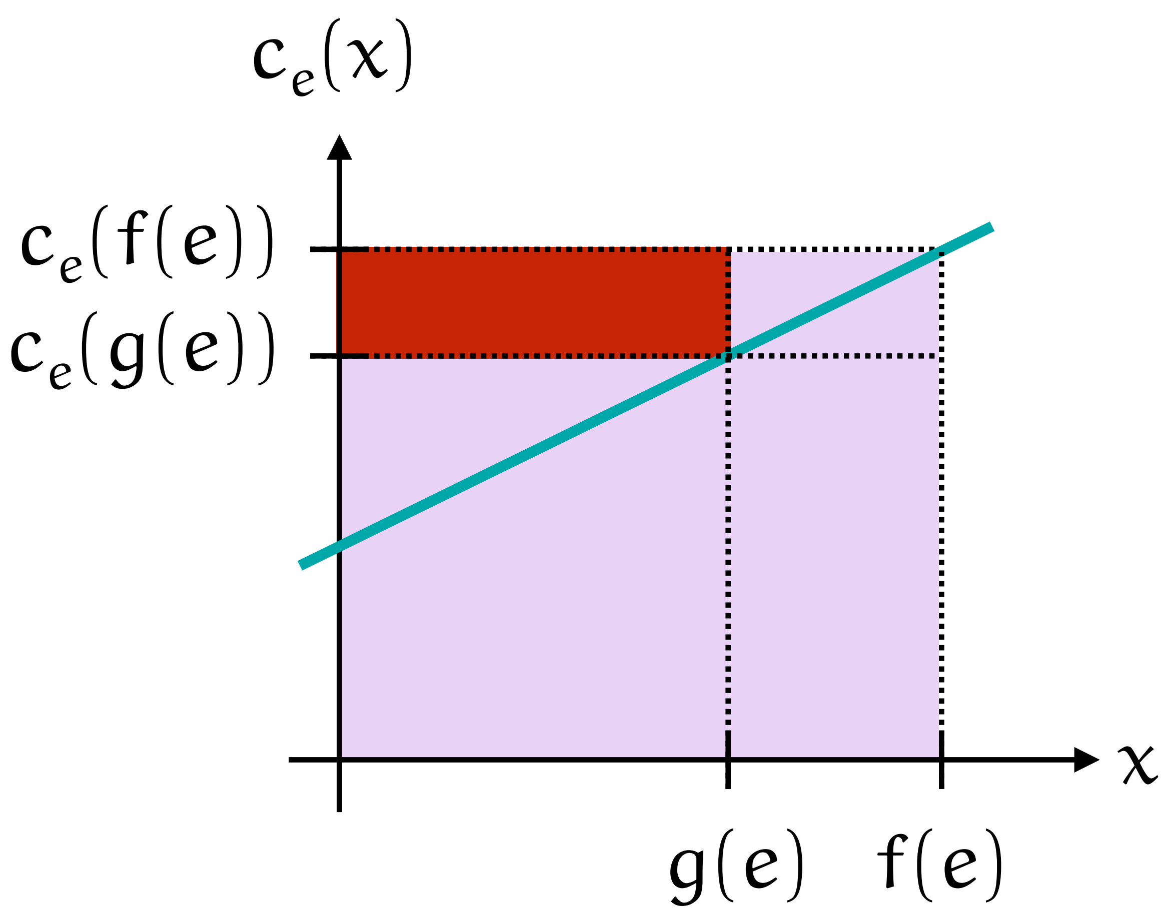
  $\leqslant \sum_{e\in E} c_e(g(e))\, g(e) + {}^1\!/_4 \sum_{e\in E} c_e(f(e))\, f(e) = C(\mathbf{g}) + {}^1\!/_4\, C(\mathbf{f})$



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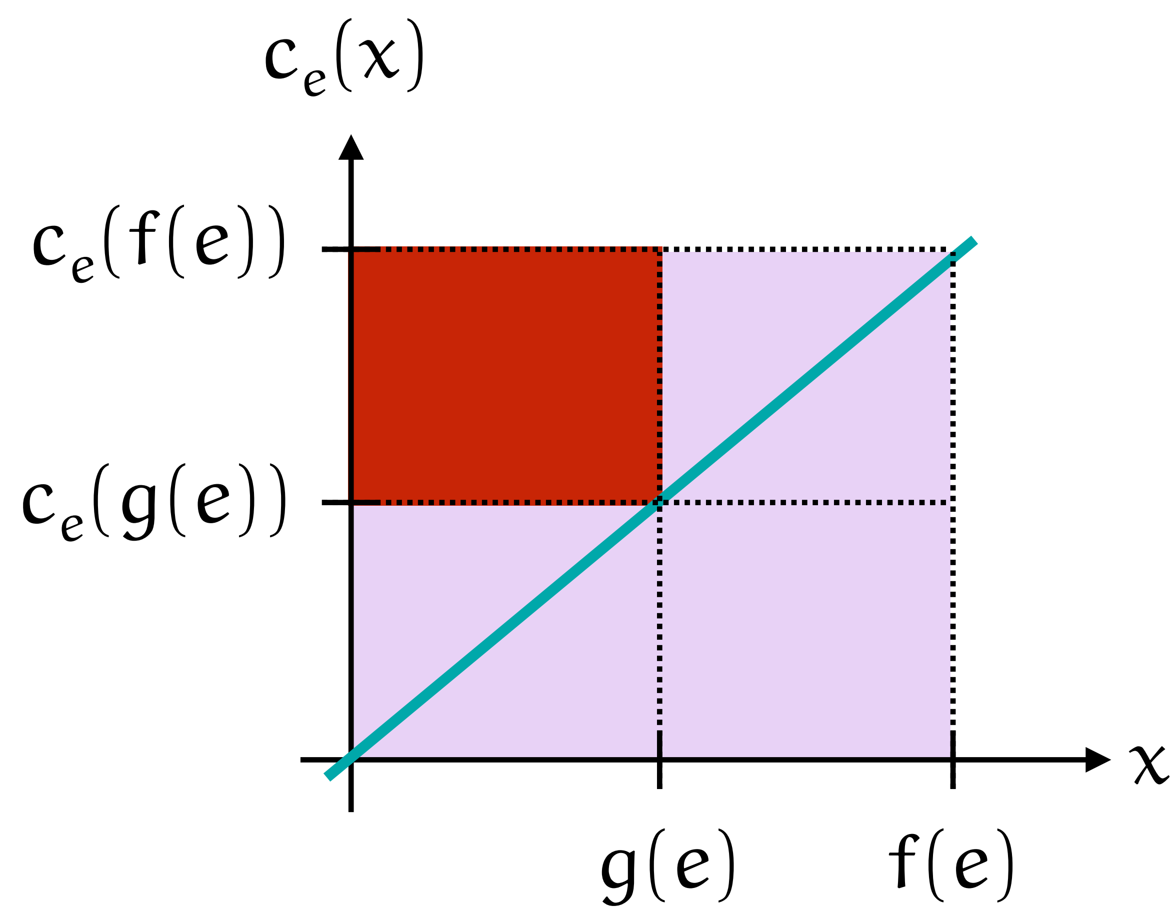
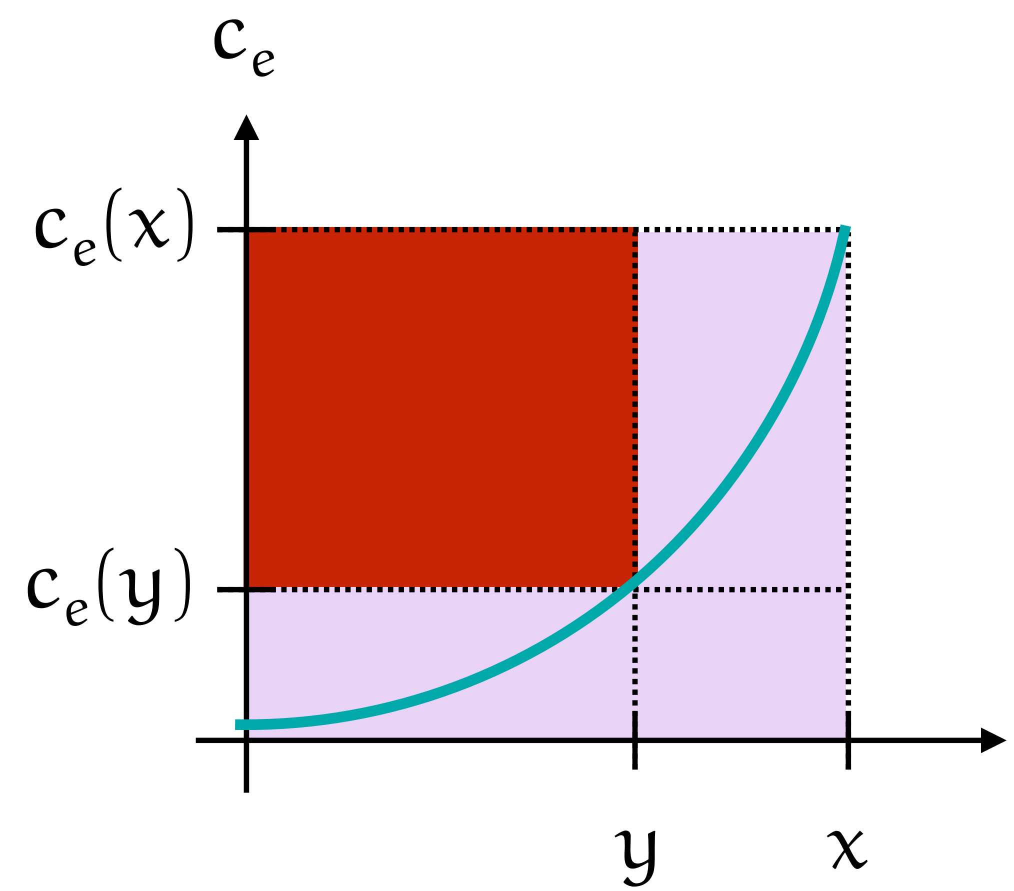


Bound for arbitrary costs

- for an arbitrary set $\mathcal{C}$ of cost functions let

$$\beta(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{x,y \in \mathbb{R}_+} \frac{(c(x) - c(y))y}{c(x)x}$$



Theorem [Roughgarden '03]

$\mathrm{PoA} \leqslant (1-\beta(\mathcal{C}))^{-1}$ for all networks with costs from the set $\mathcal{C}$.

- gives $4/3$ for affine functions , quadratic functions →Exercise session
- closed formula for polynomials, BPR functions, and MM1 functions
- unbounded for general functions

Unsplittable flows

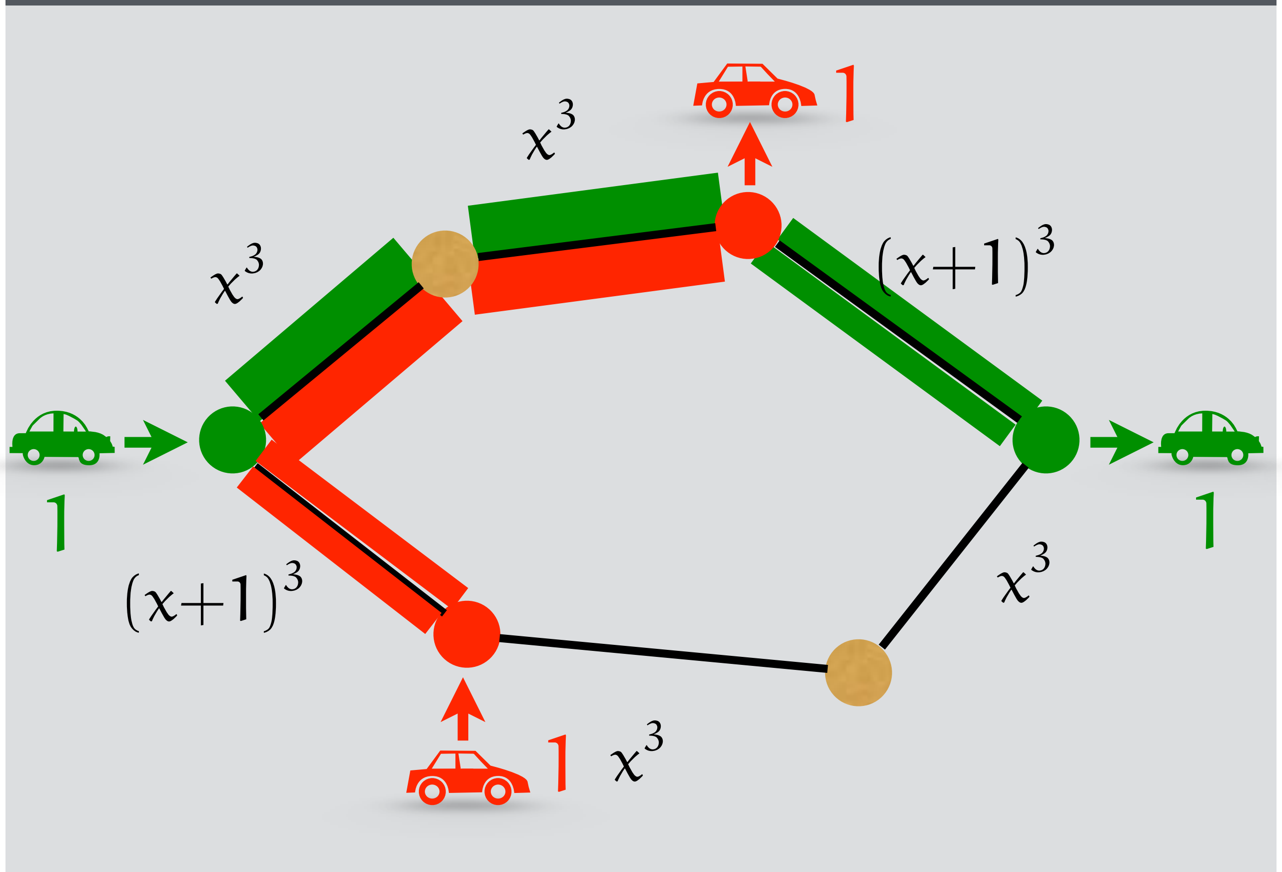
Introduction

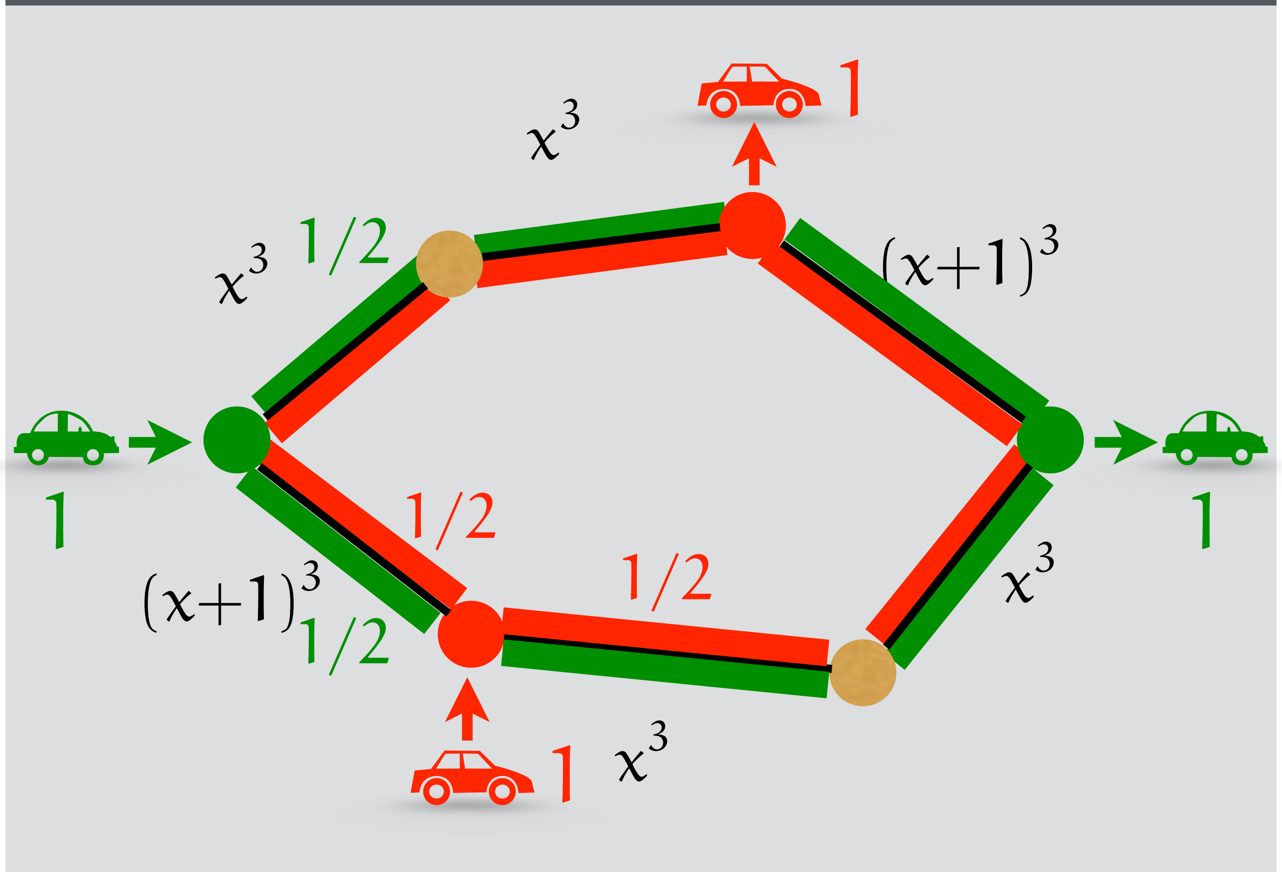
Critique of non-atomic models

- non-atomic models assume that each commodity consists of a large population of infinitesimally small players, each with negligible impact
- population of a commodity may split arbitrarily between the paths in a network
- unrealistic in telecommunication applications where all data is send along a single path under current TCP/IP protocol
  (to ensure that packets arrive in order)

Atomic vs. non-atomic games

Atomic	Non-atomic



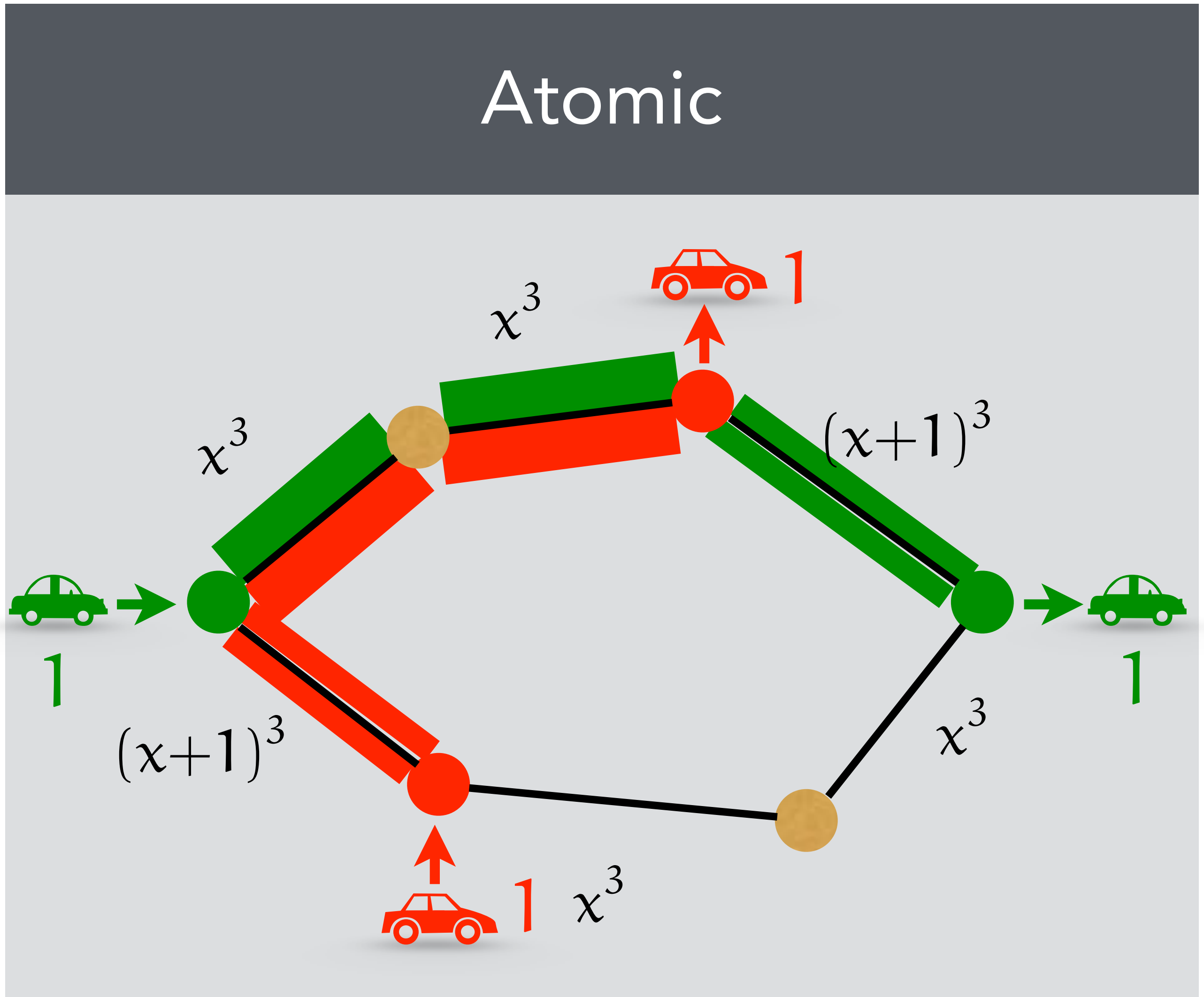


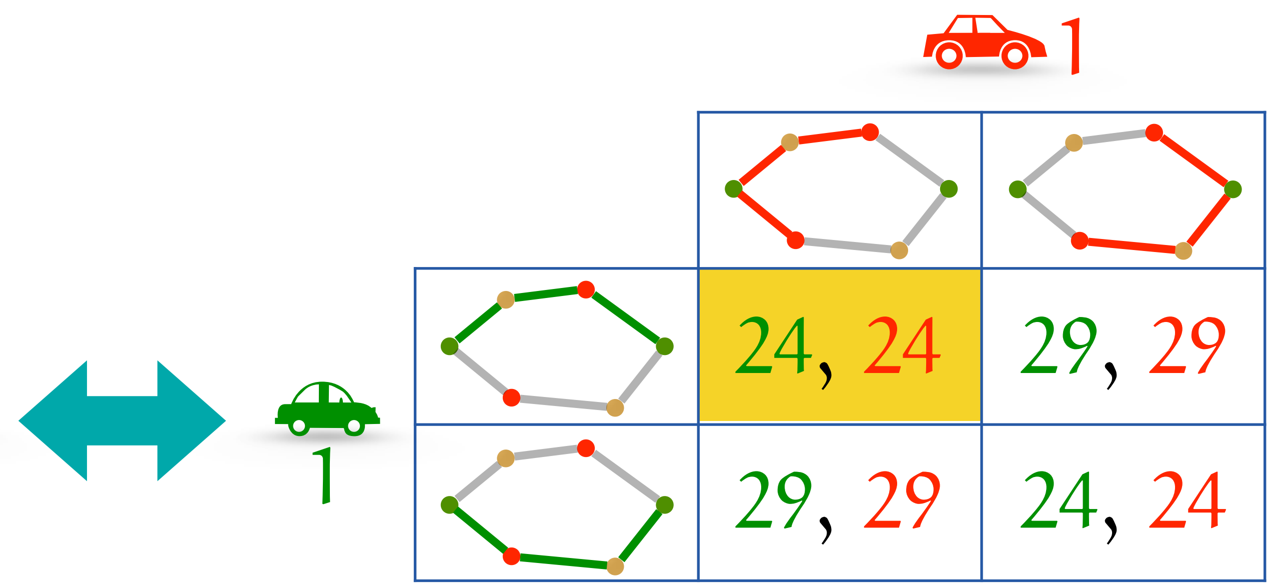
- commodities do not split
- every commodity corresponds to an individual player

- commodities split arbitrarily
- every flow particle corresponds to an individual player

Limit when number of players increase and their weight decreases

Atomic games as strategic games



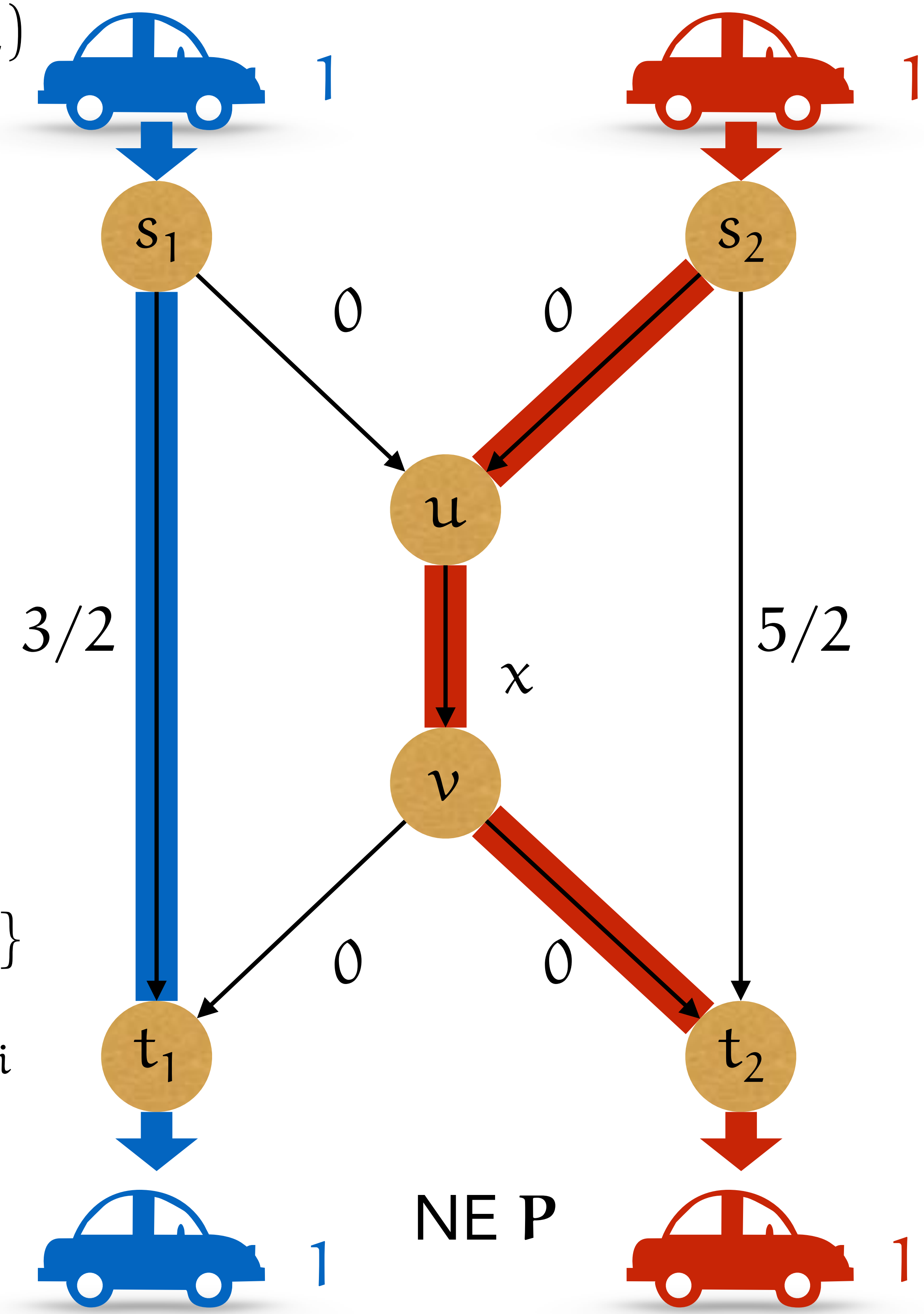


- Atomic congestion games are finite strategic games
  - finite set of players
  - each player has a finite set of strategies

Formal model

- directed or undirected graph $G = (V, E)$
  - finite set of vertices V
  - set of edges $E \subseteq V \times V$
- cost function $c_e : \mathbb{R}_+ \to \mathbb{R}_+$ for $e \in E$
  - continuous
- set $N = \{1, \dots, n\}$ of players, each with
  - origin vertex $s_i \in V$
  - destination vertex $t_i \in V$
  - demand $d_i \in \mathbb{R}_+$
  - strategy set $\mathcal{P}_i = \{P : P \text{ is } (s_i, t_i)\text{-path}\}$
  - private cost for $\mathbf{P} = (P_1, \dots, P_n)$; $P_i \in \mathcal{P}_i$

$$\pi_i(\mathbf{P}) = \sum_{e \in P_i} c_e\Big(\underbrace{\sum_{j \in N \,:\, e \in P_j} d_j}_{f_e(\mathbf{P})}\Big)$$



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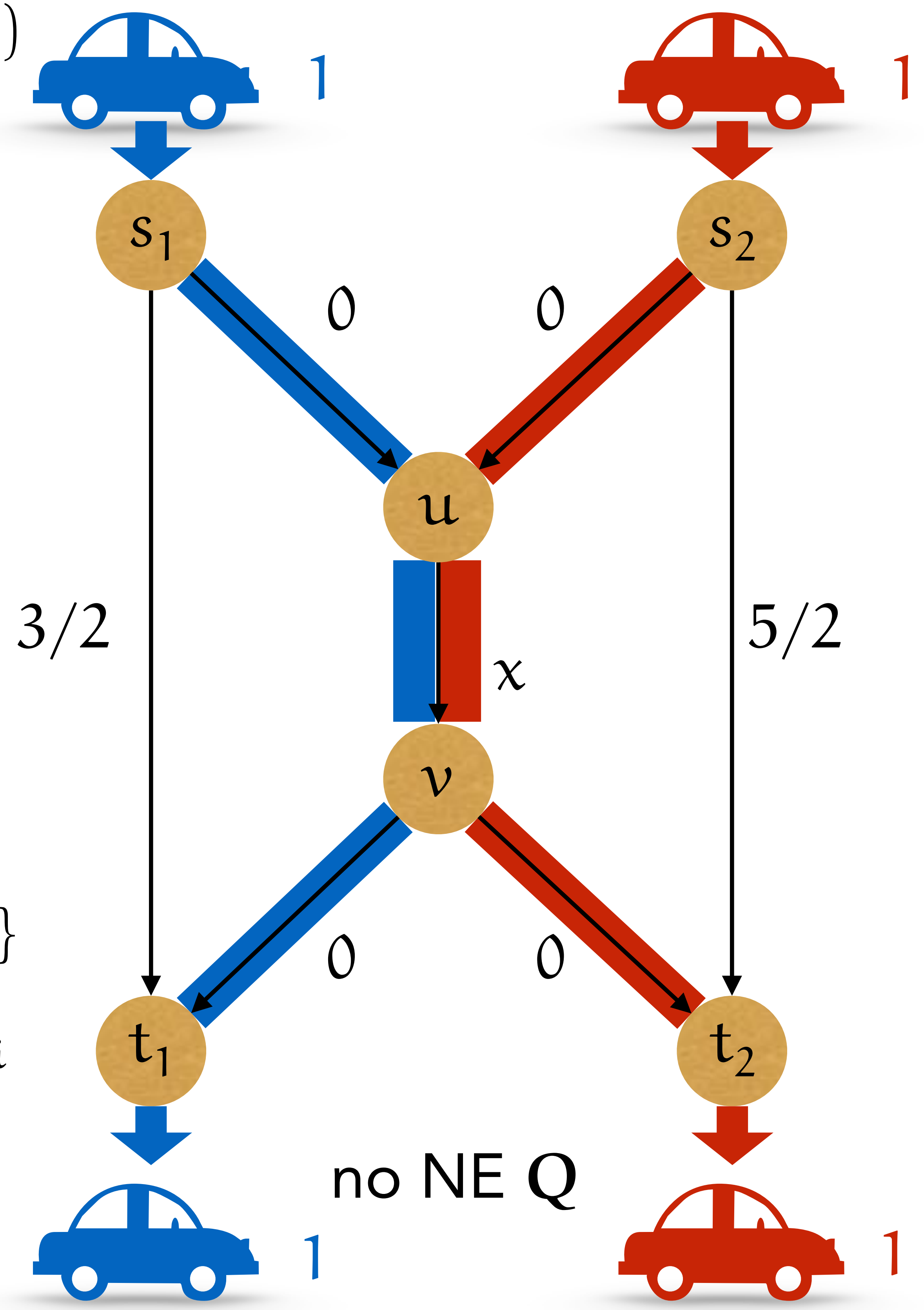


Equilibria

Definition — (pure) Nash equilibrium

path profile $\mathbf{P}$ such that

$\pi_i(Q_i, \mathbf{P}_{-i}) \geqslant \pi_i(P_i, \mathbf{P}_{-i}) \quad \forall i \in N,\ Q_i \in \mathcal{P}_i$

- mixed strategy x_i of player i

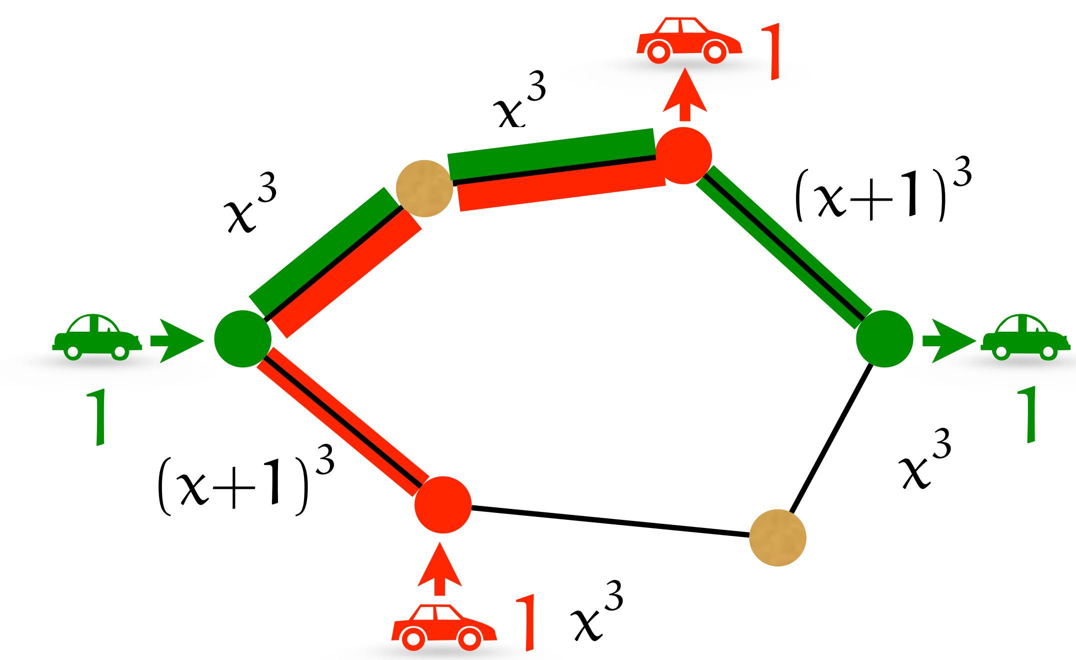
  is a probability distribution over $\mathcal{P}_i$

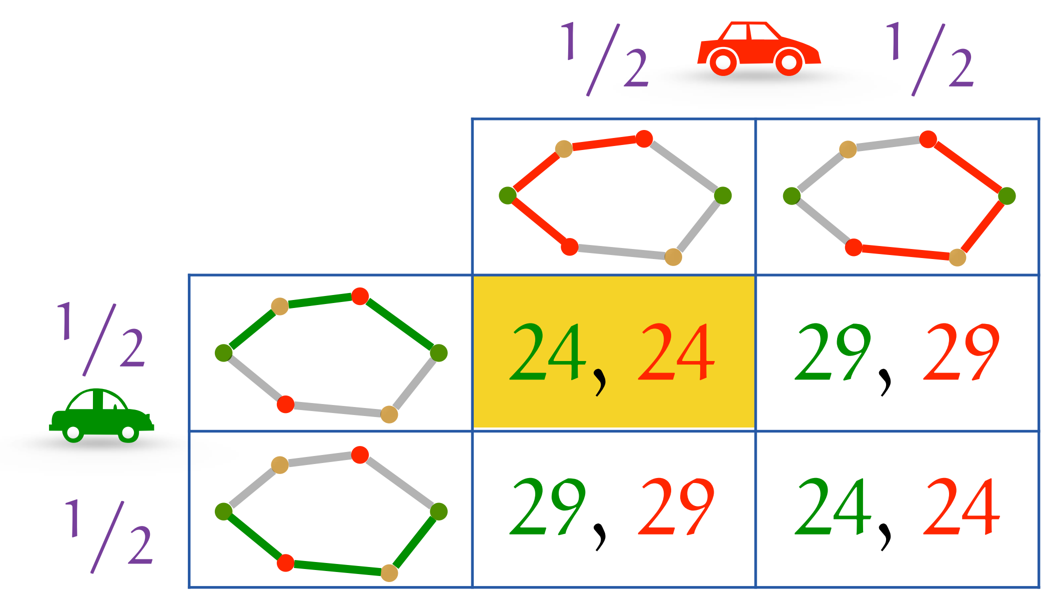
  $x_i = (x_{i,P_1}, x_{i,P_2}, \dots) \in \Delta(\mathcal{P}_i)$
- expected private costs

  $\bar{\pi}_i(x_i, \mathbf{x}_{-i}) = \mathbb{E}_{\mathbf{x}}[\ \pi_i(P_i, \mathbf{P}_{-i})\ ]$
- $\bar{\pi}_i(x_i, \mathbf{x}_{-i}) = \sum_{P \in \mathcal{P}_i} x_{i,P} \cdot$

  $\sum_{e \in P} \mathbb{E}[c_e(d_i + f_{e,-i}(\mathbf{P}_{-i})],$

  where $f_{e,-i}(\mathbf{P}_{-i}) = \sum_{j \in N \setminus \{i\} :\, e \in P_j} d_j$





Definition — mixed Nash equilibrium

prob. dist. profile $\mathbf{x}$ such that

$\pi_i(y_i, \mathbf{x}_{-i}) \geqslant \pi_i(y_i, \mathbf{x}_{-i}) \quad \forall i \in N,\ y_i \in \Delta(\mathcal{P}_i)$

mixed equilibrium

Unsplittable flows

Existence of equilibria

Congestion Games

Theorem [Nash `52]

Every finite game has a mixed NE.

- proof via Brouwer's fixed point theorem
- pure NE need not exist:

mixed equilibrium

		$^1/_3$	$^1/_3$	$^1/_3$
$^1/_3$		0, 0	−1, 1	1, −1
$^1/_3$		1, −1	0, 0	−1, 1
$^1/_3$		−1, 1	1, −1	0, 0

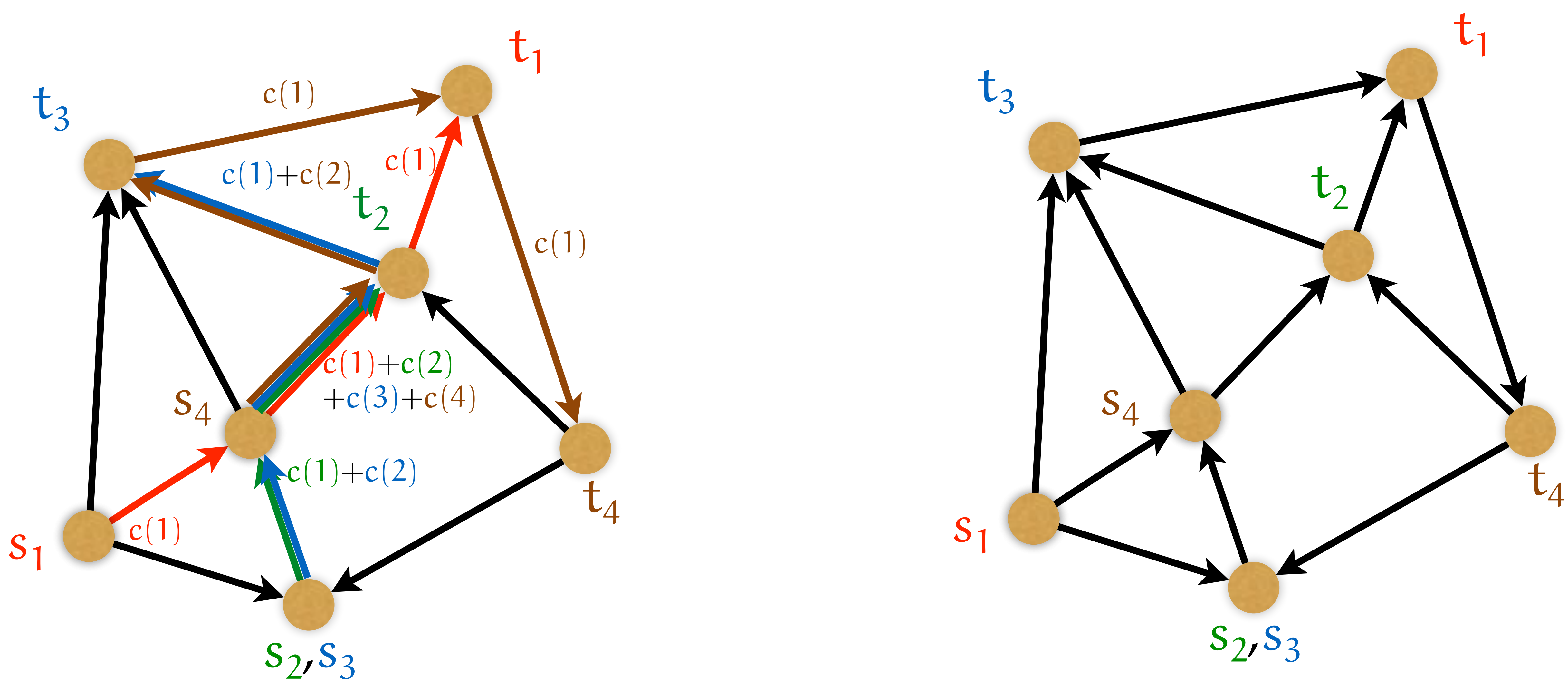
Theorem [Rosenthal `73]

Every unweighted congestion game ($d_i = 1 \ \forall i$) has a pure Nash equilibrium.

- proof via potential functions

The potential function argument

- let $\Phi(\mathbf{P}) = \sum_{e \in E} \Phi_e(\mathbf{P})$, where $\Phi_e(\mathbf{P}) = \sum_{k=1,\dots,f_e(\mathbf{P})} c_e(k)$



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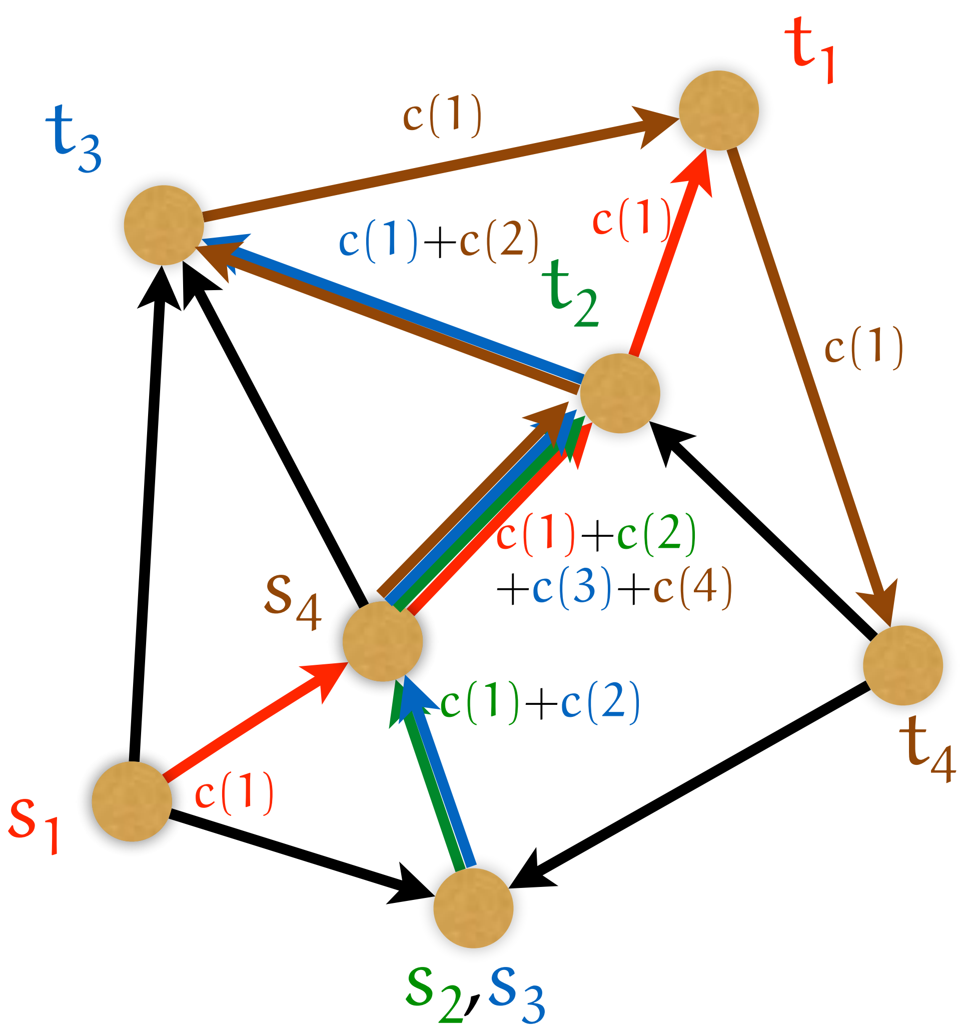

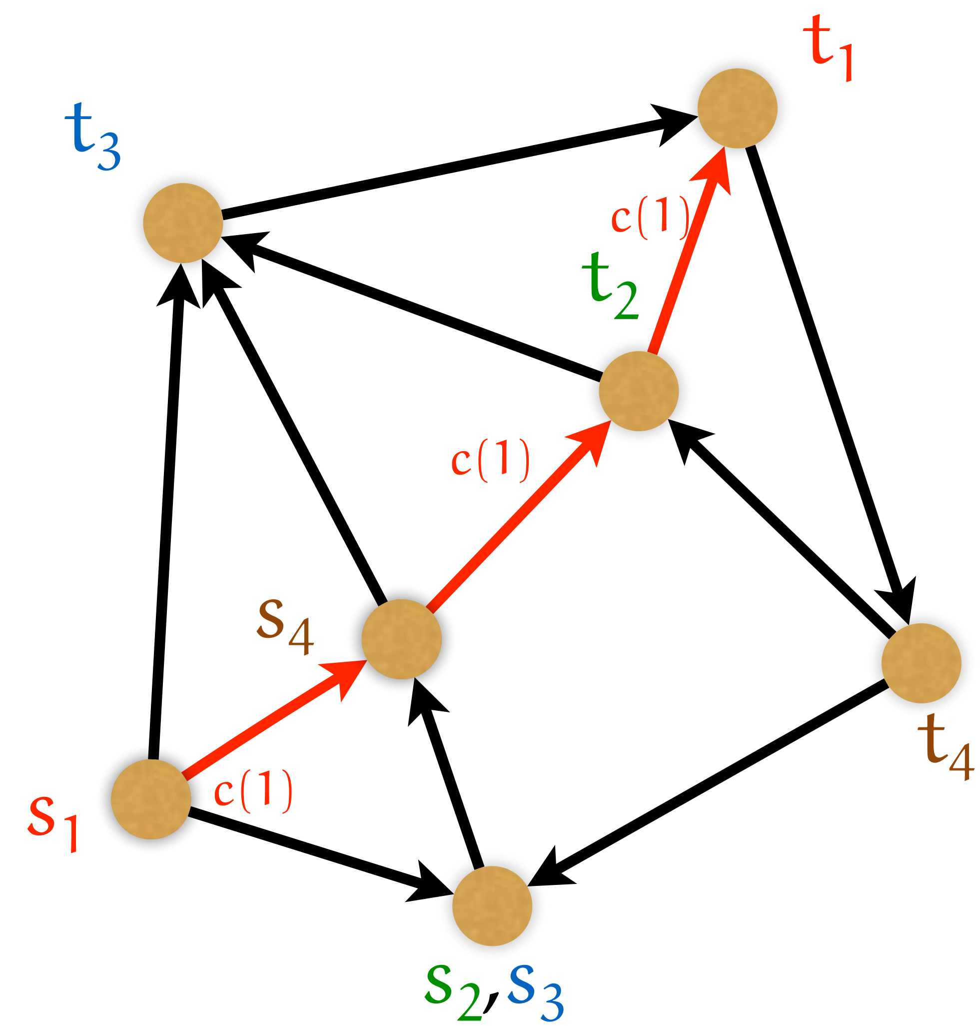

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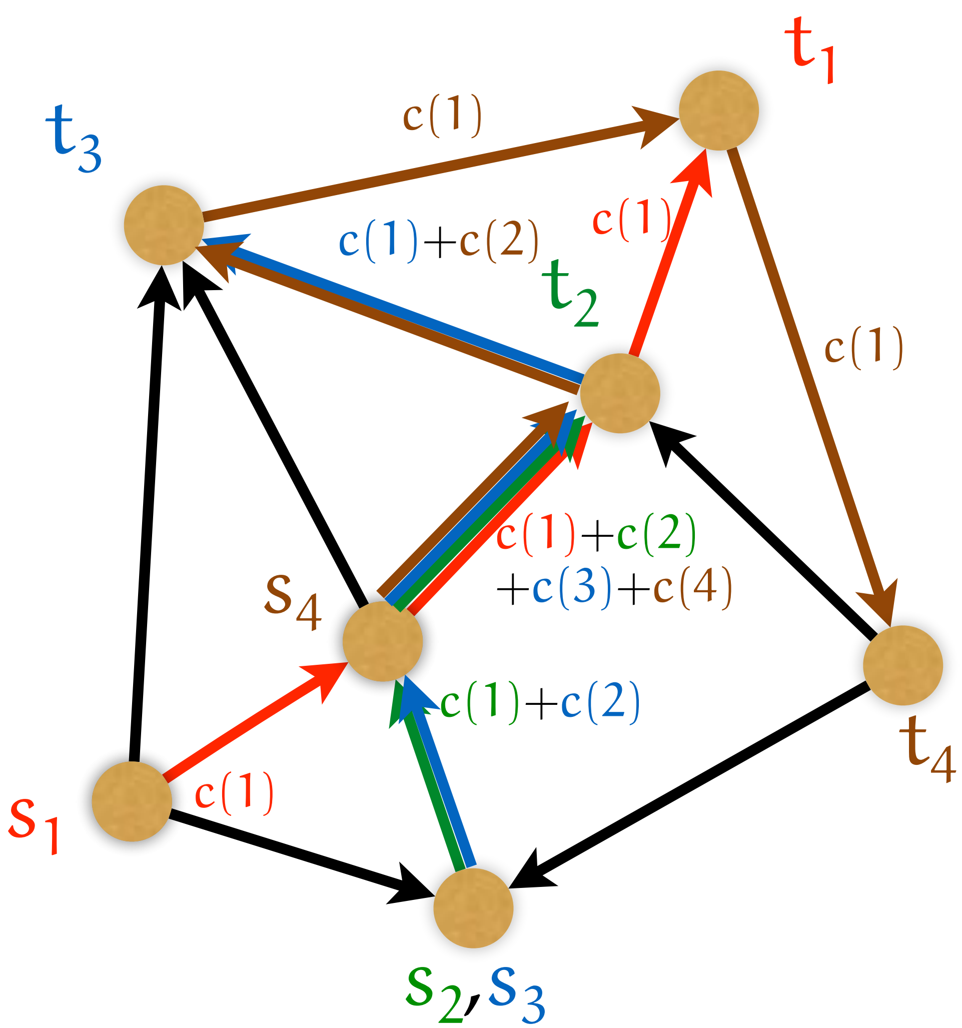

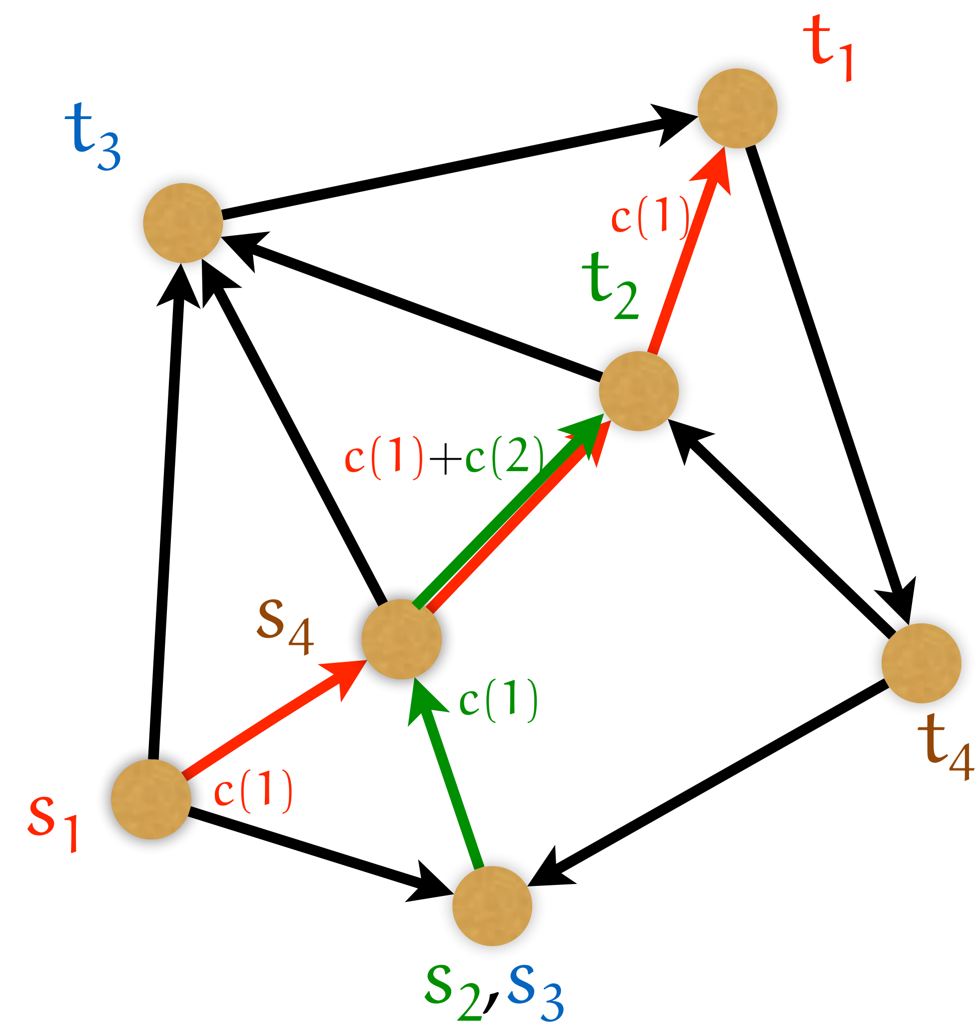

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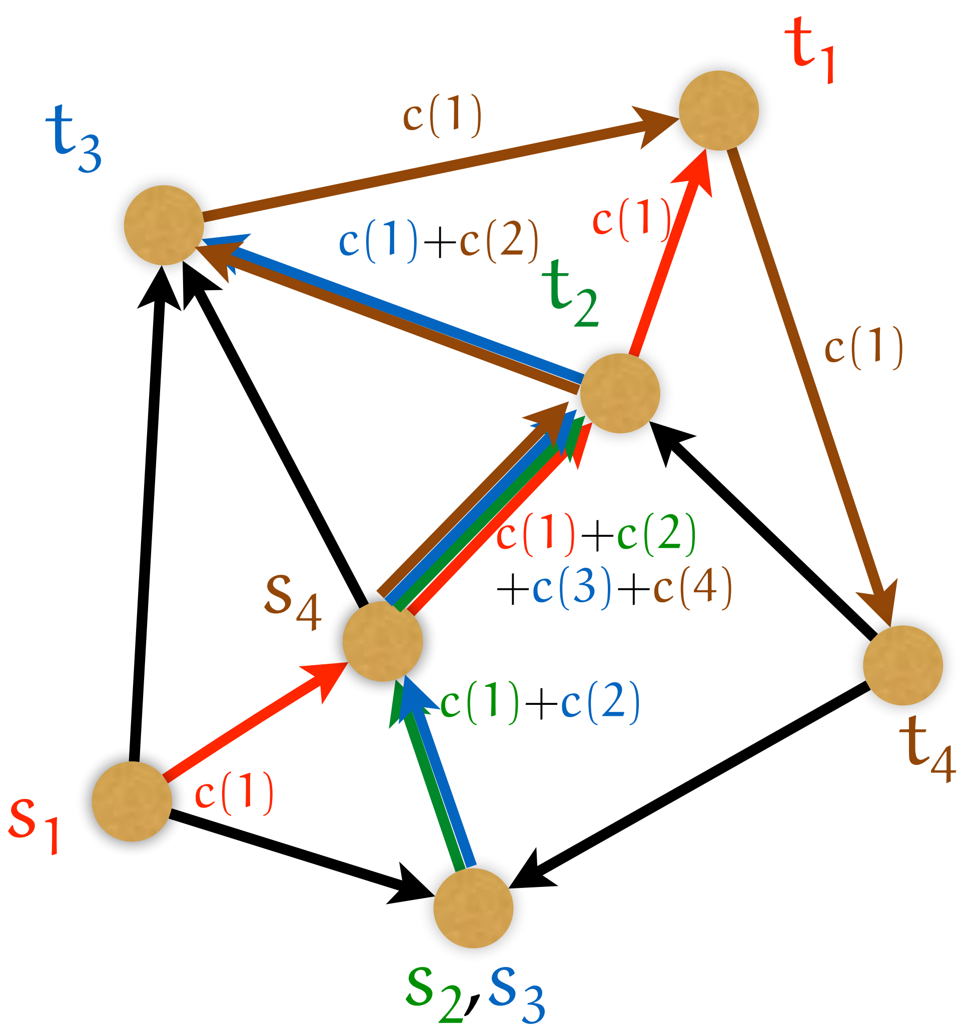

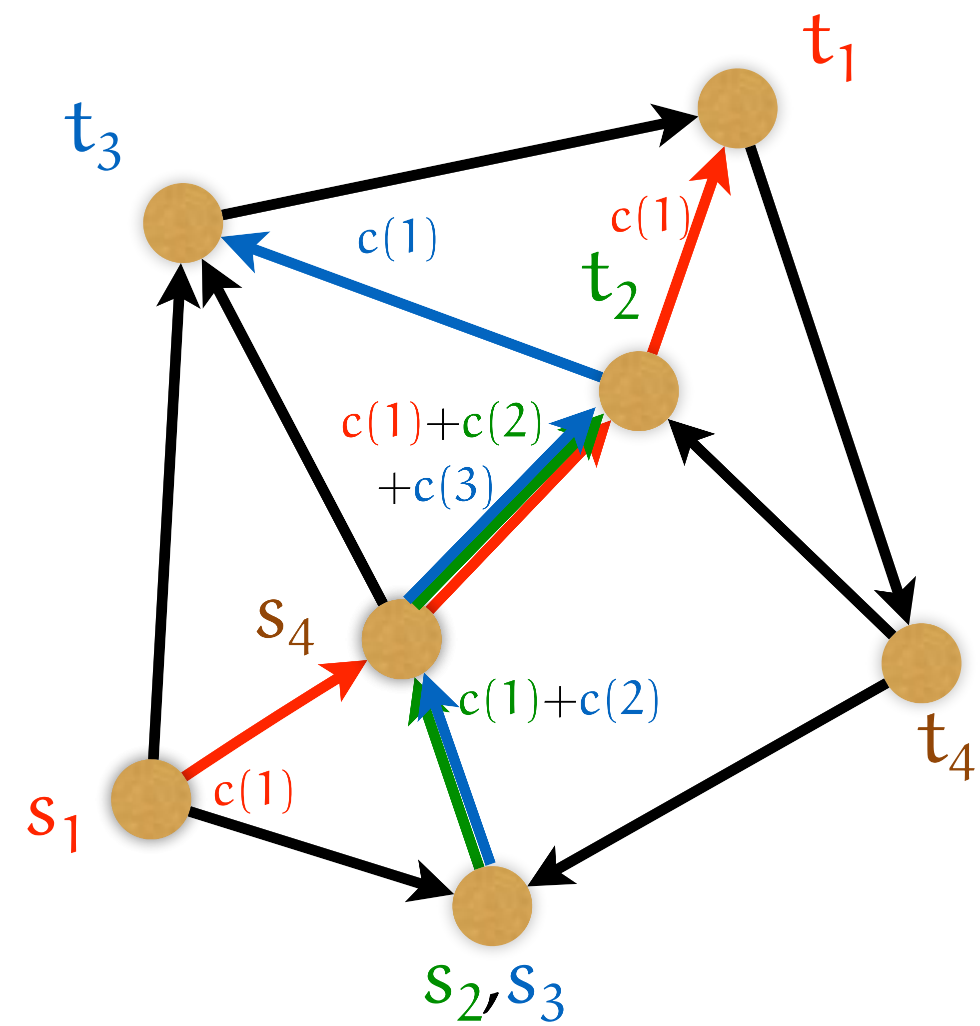

The potential function argument

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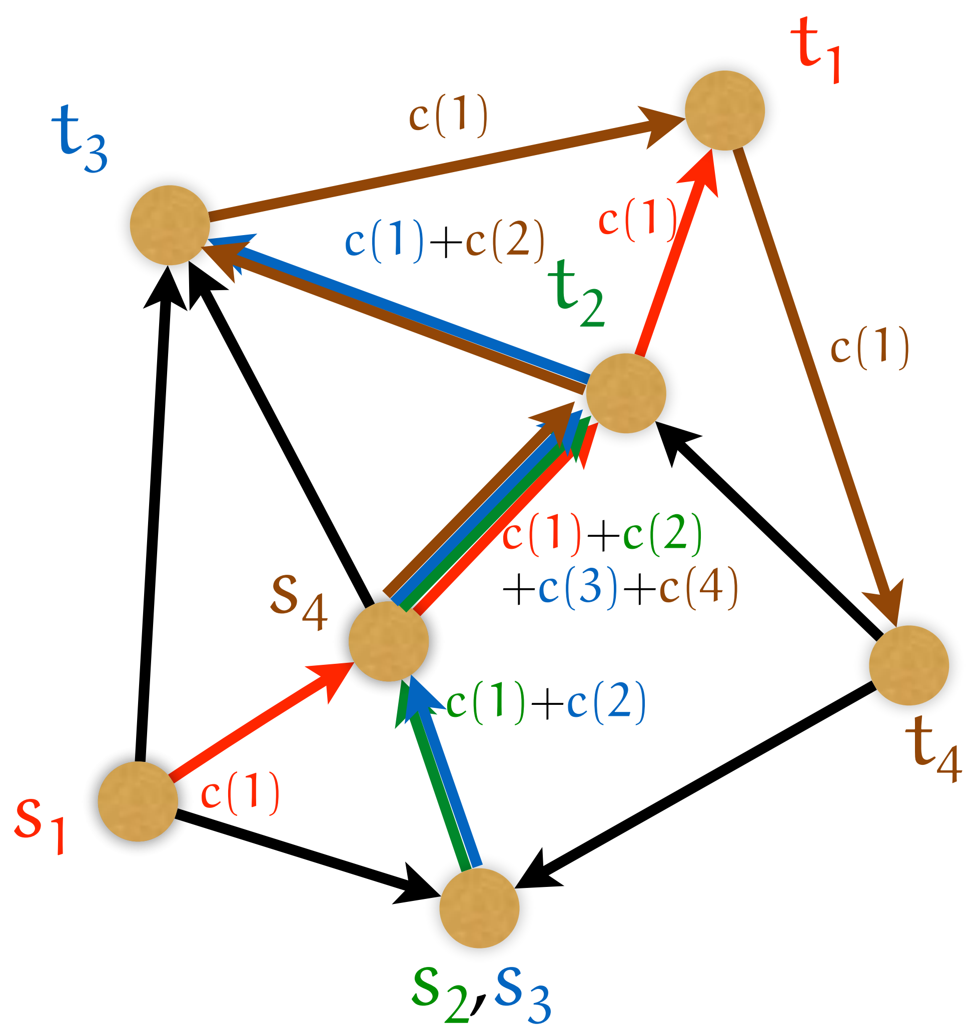

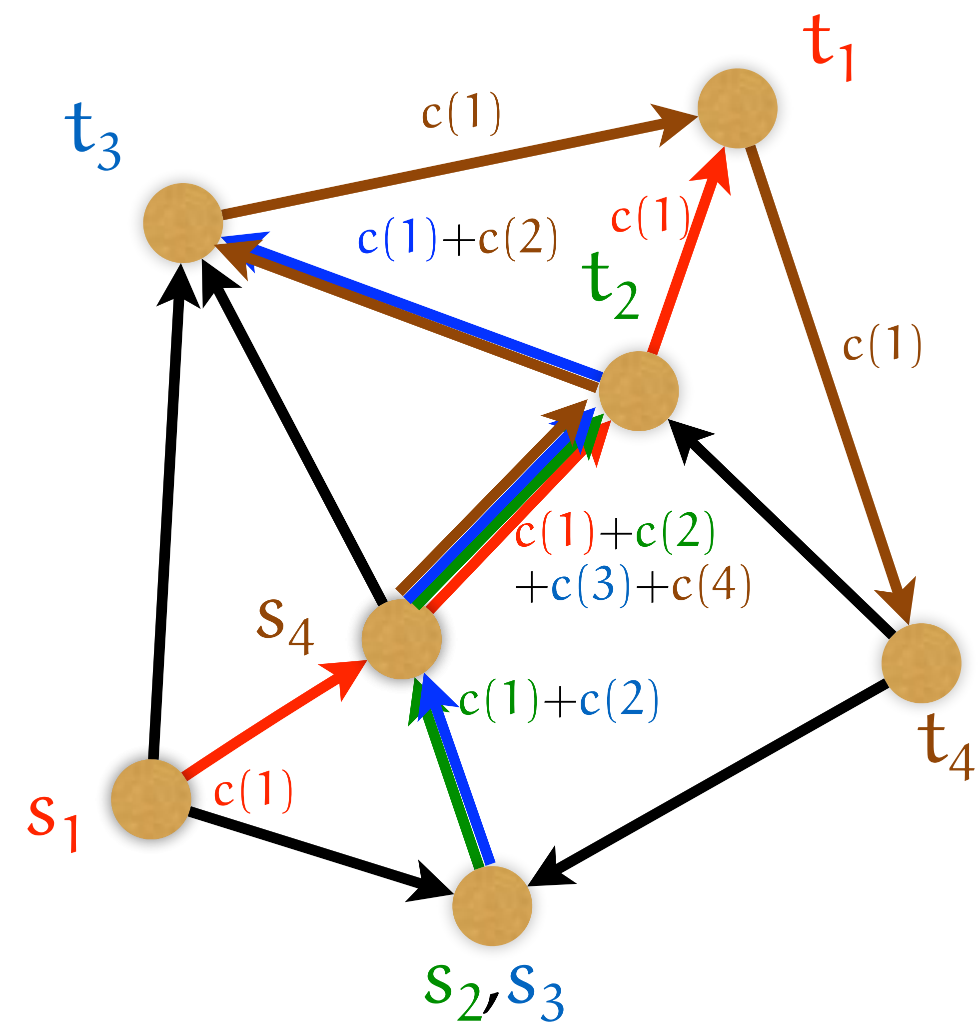

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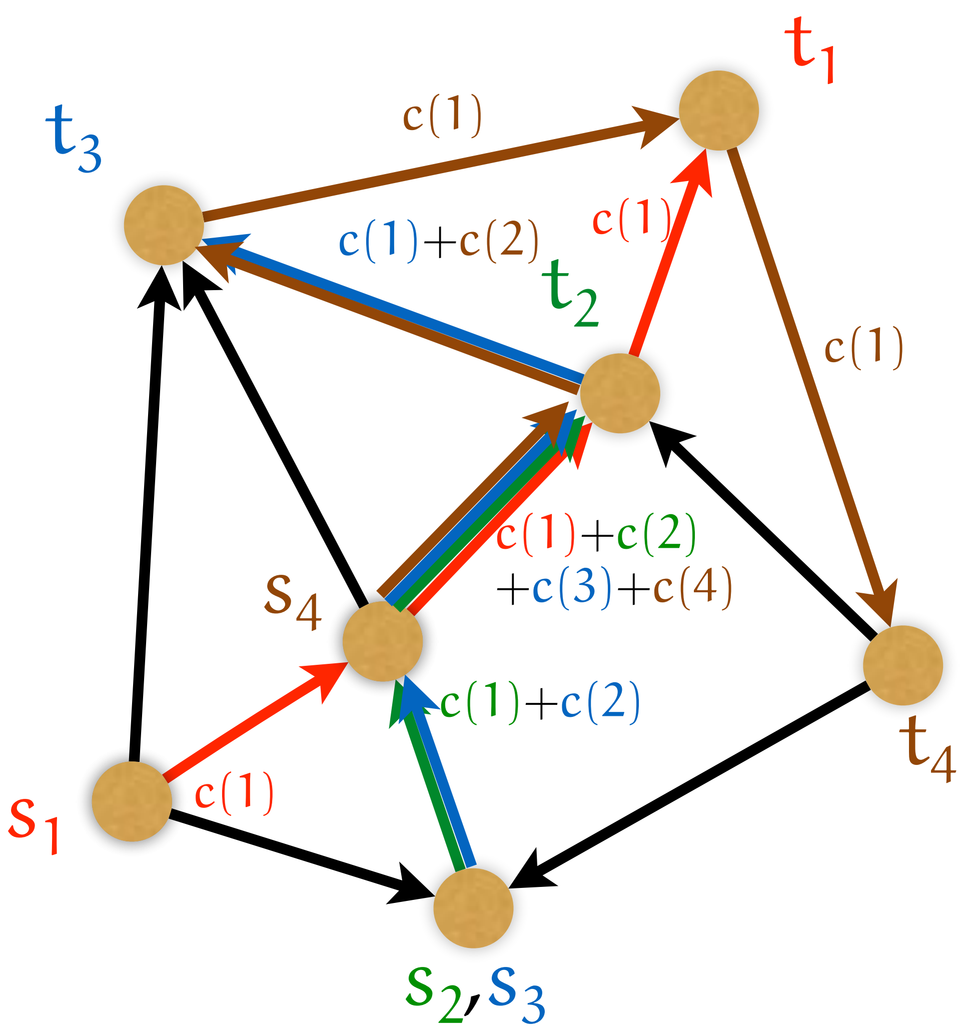
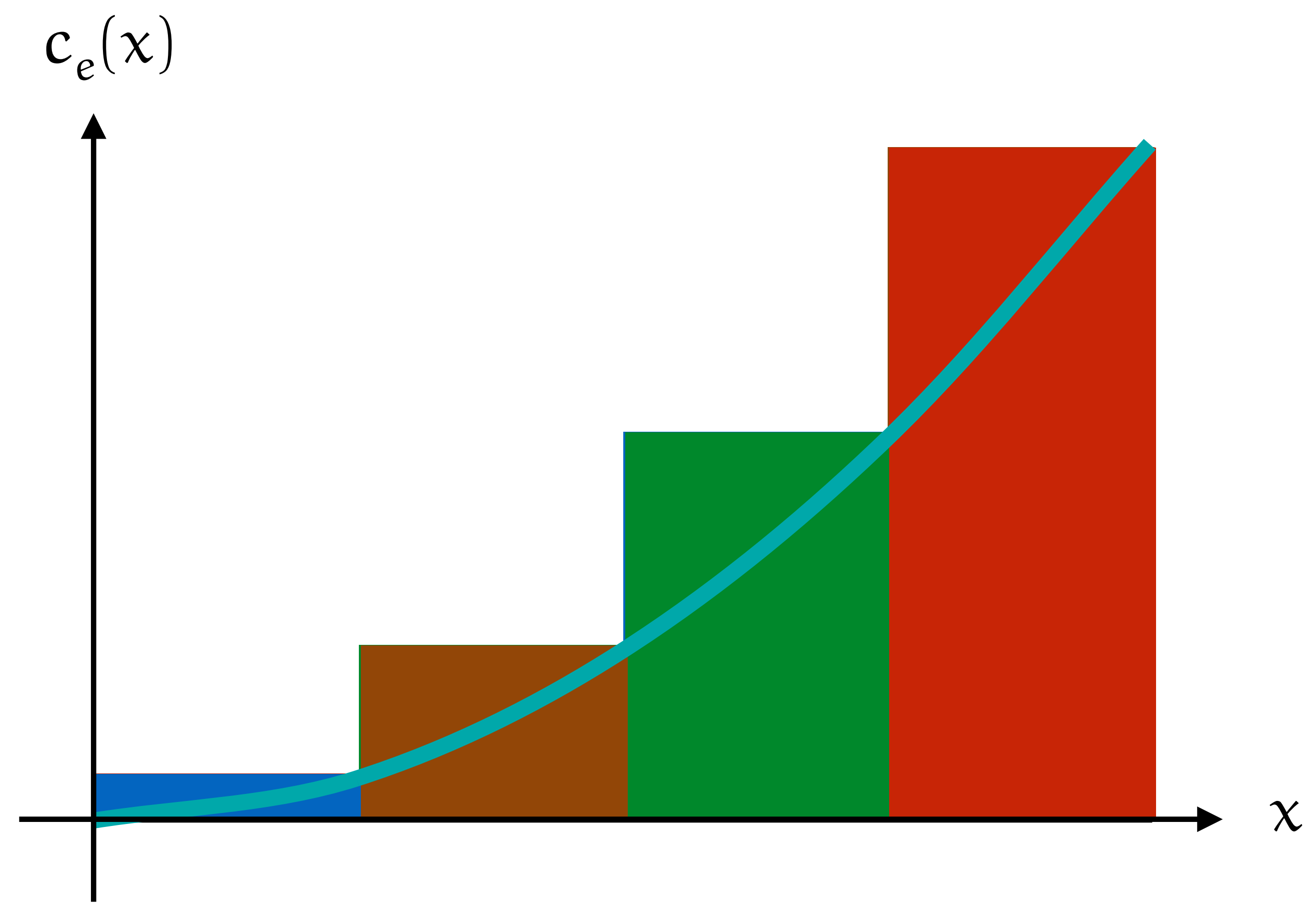
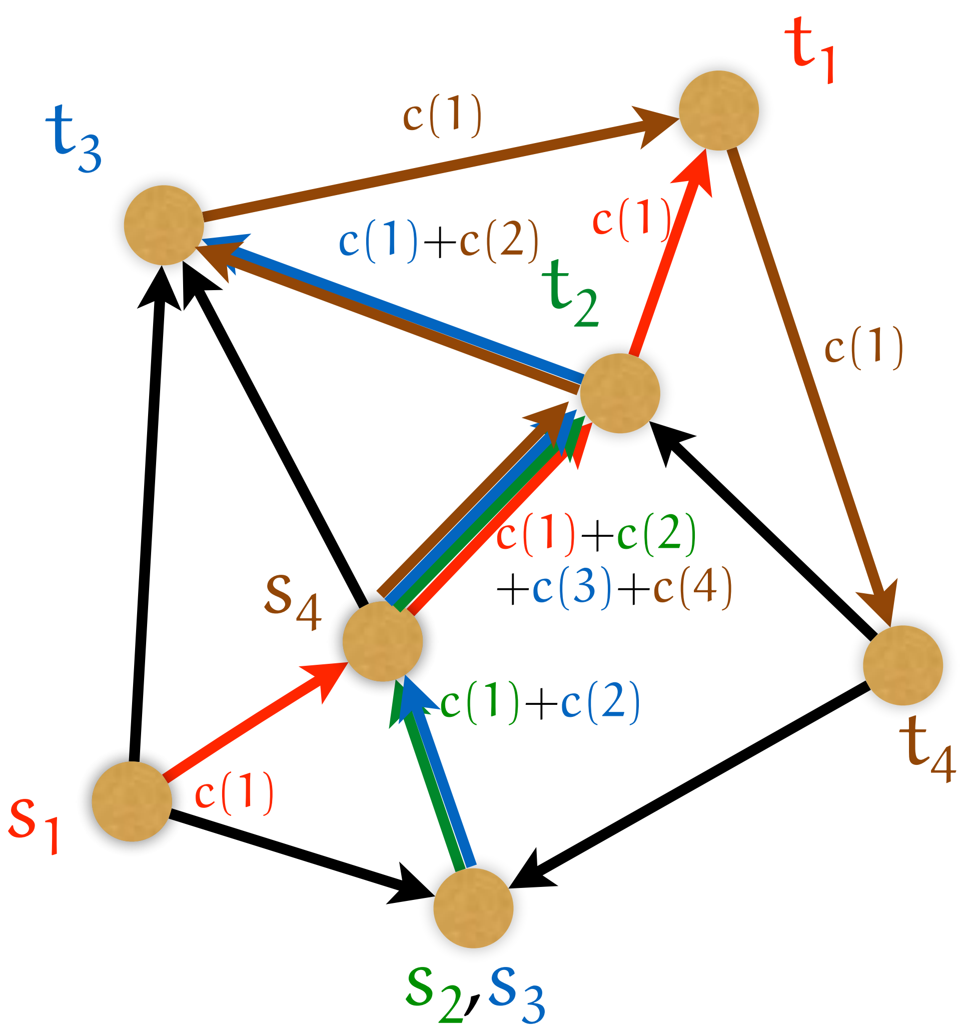




Observation: Potential function independent of ordering of the players

The potential function argument



Theorem [Rosenthal `73]

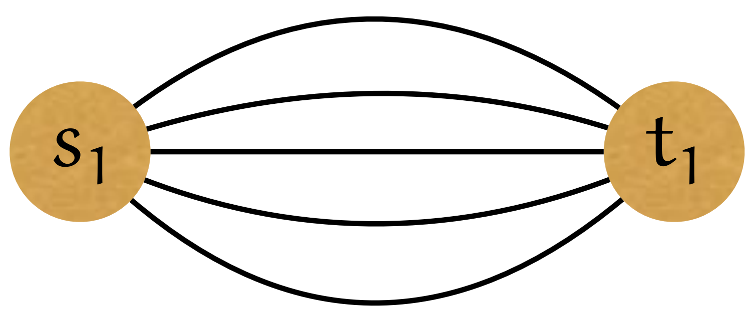
Every unweighted congestion game has a pure Nash equilibrium.

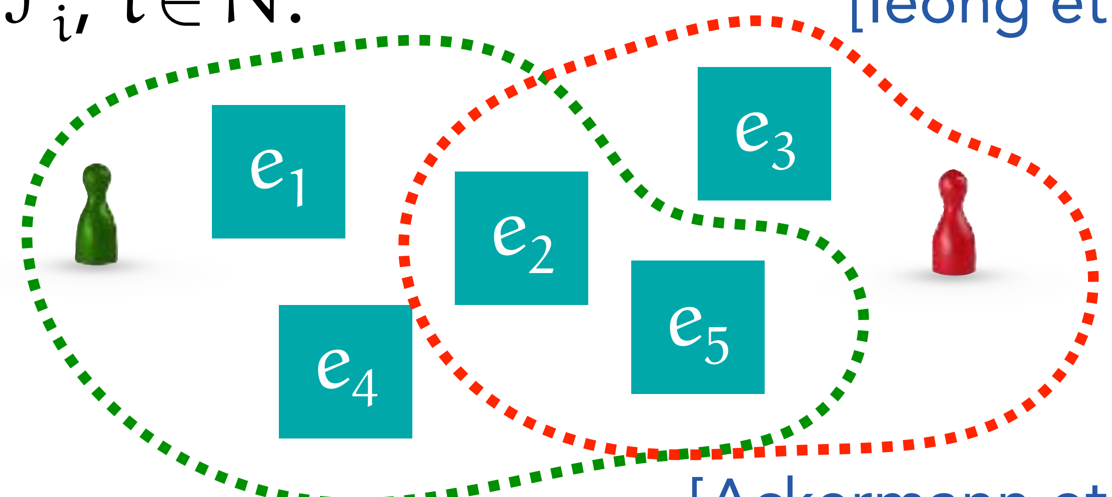
Proof

- consider profitable deviation of n from $\mathbf{P} = (P_n, \mathbf{P}_{-n})$ to $\mathbf{Q} = (Q_n, \mathbf{P}_{-n})$
- $\Phi(\mathbf{Q}) - \Phi(\mathbf{P})$
  $= \sum_{e \in Q_n} c_e(f_e(\mathbf{Q})) - \sum_{e \in P_n} c_e(f_e(\mathbf{P}))$
  $= \pi_n(\mathbf{Q}) - \pi_n(\mathbf{P}) < 0$
- every sequence of profitable deviations is finite
- reaches pure Nash equilibrium

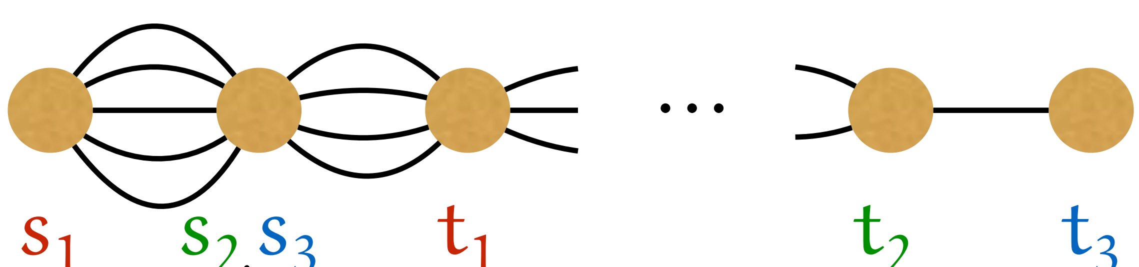
Computation of equilibria

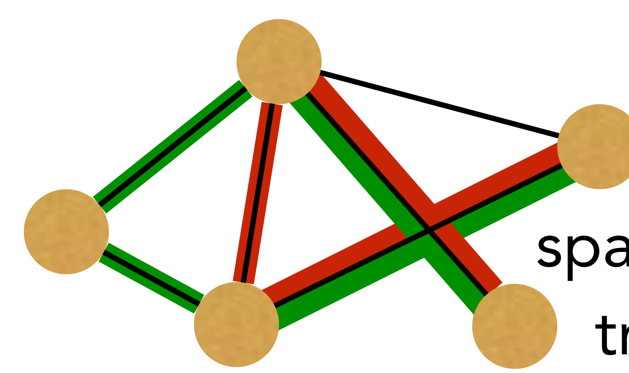
- every sequence of profitable deviations is finite
- but: convergence may take exponential time
  - computation of a pure Nash equilibrium is PLS-complete
    (as hard as any local search problem) [Fabrikant et al. '03], [Ackermann et al. '08]
- convergence is quick for special strategy spaces
  - singletons, i.e. $|P| = 1$ for all $P \in \mathcal{P}_i$, $i \in N$. [Ieong et al. '05]





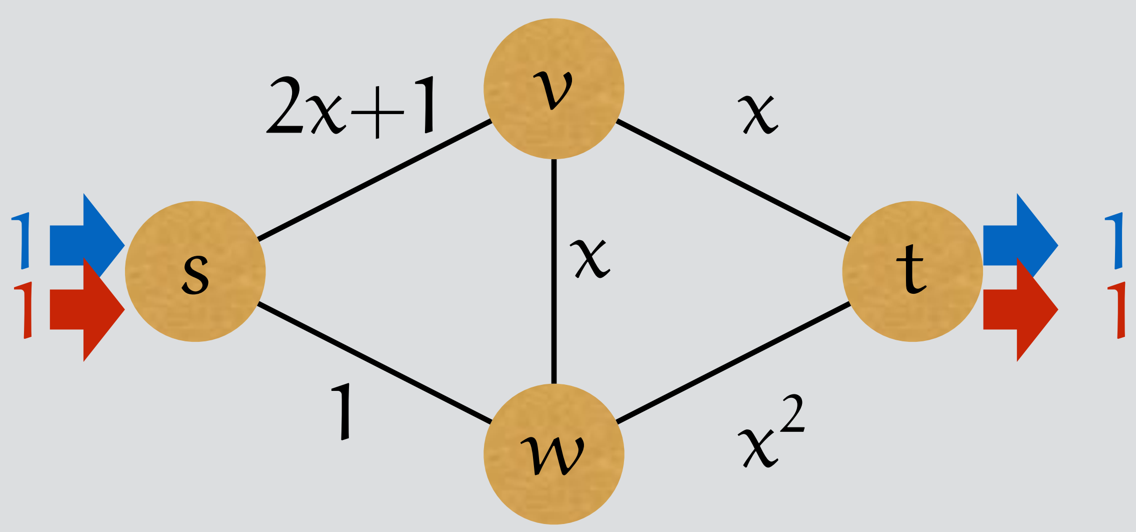
  - the basis of a matroid. [Ackermann et al. '09]



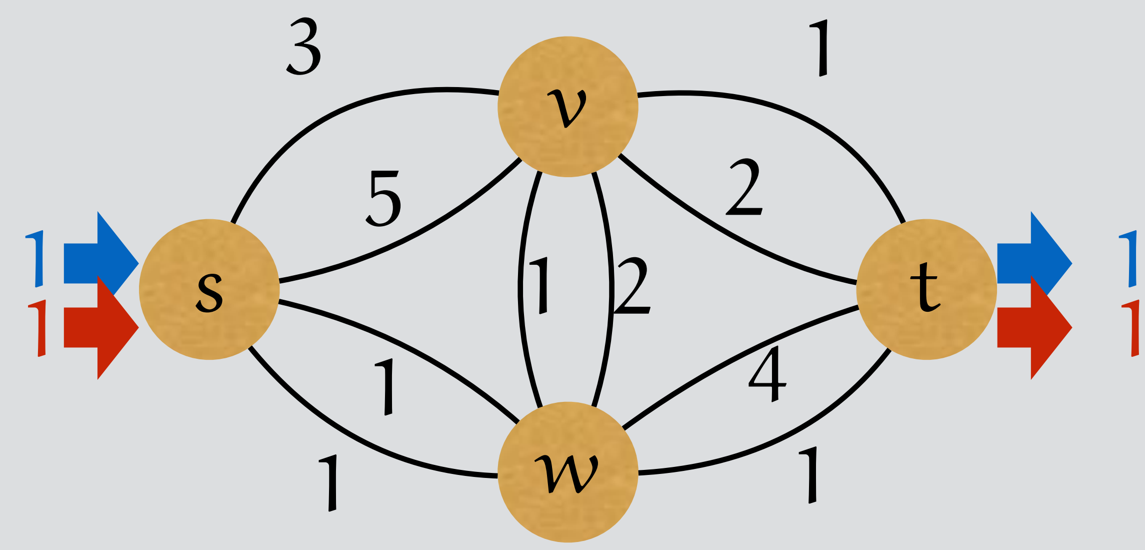


spanning trees

- for a single source and destination and non-decreasing costs the potential function can be minimized efficiently by min-cost flow computations [Fabrikant et al. '03]



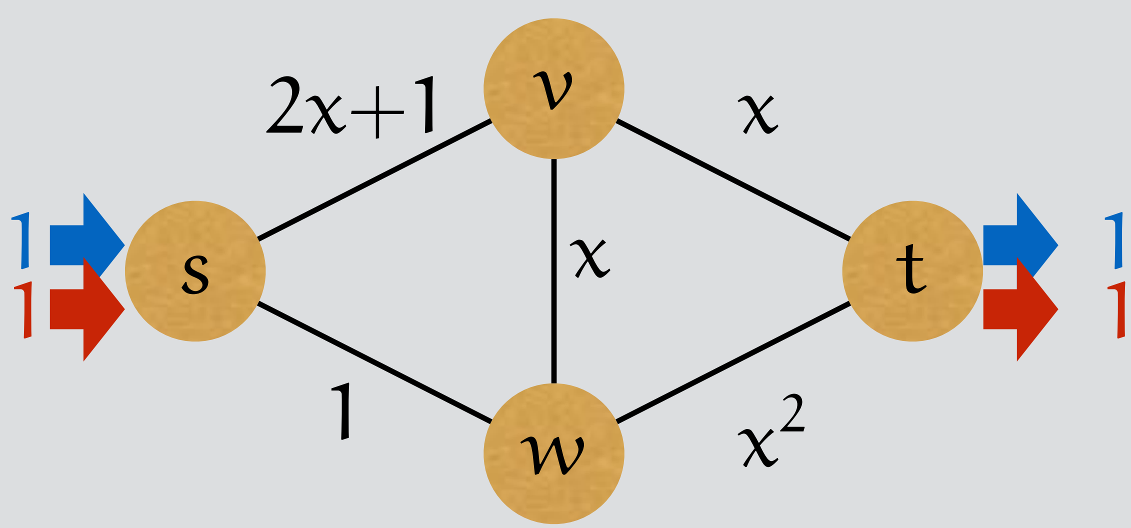
original instance



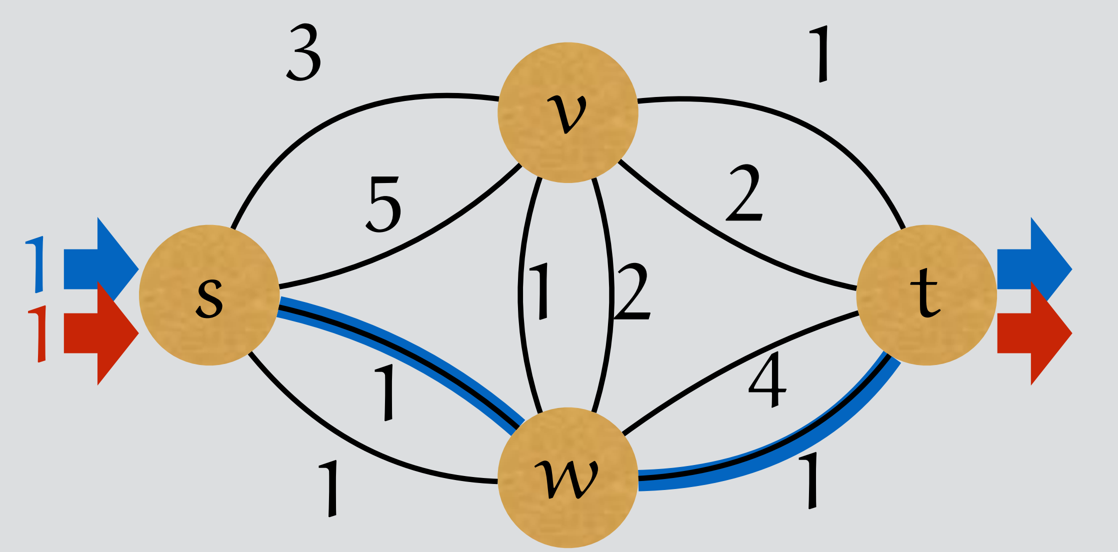
min-cost flow problem
(all edges have unit capacity)

Computation of equilibria

- for a single source and destination and non-decreasing costs the potential function can be minimized efficiently by min-cost flow computations [Fabrikant et al. '03]



original instance



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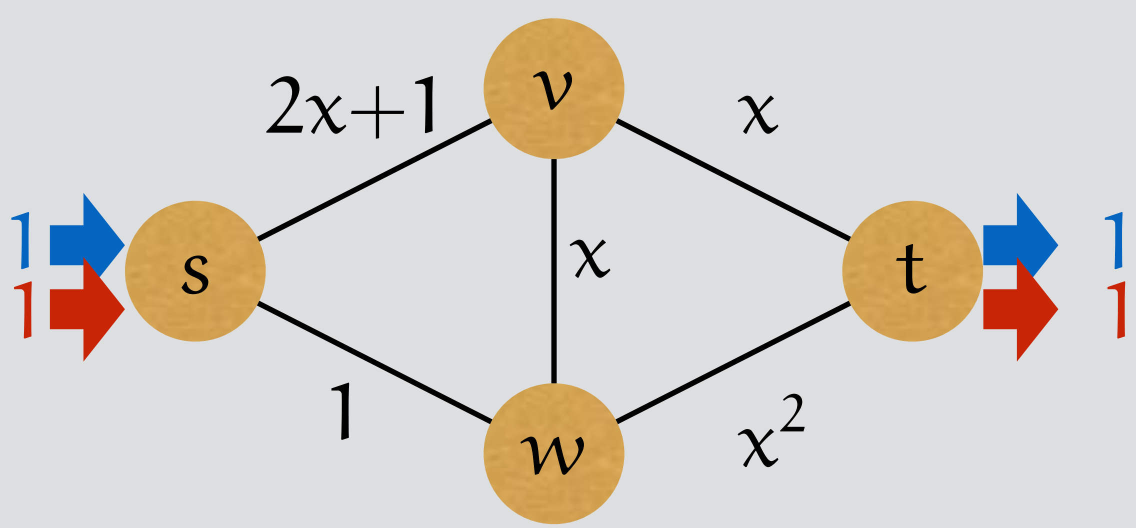
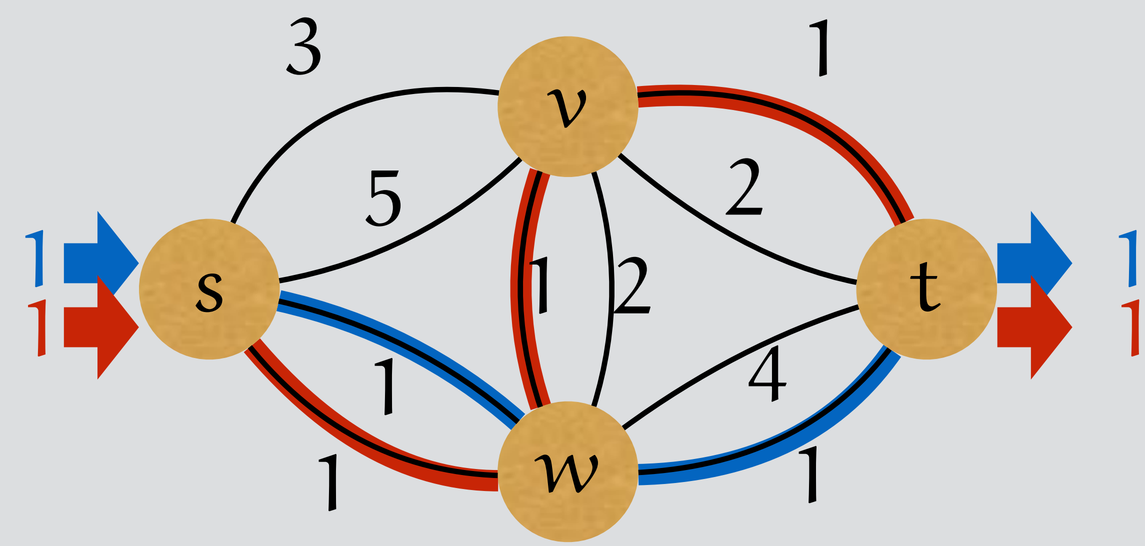


original instance

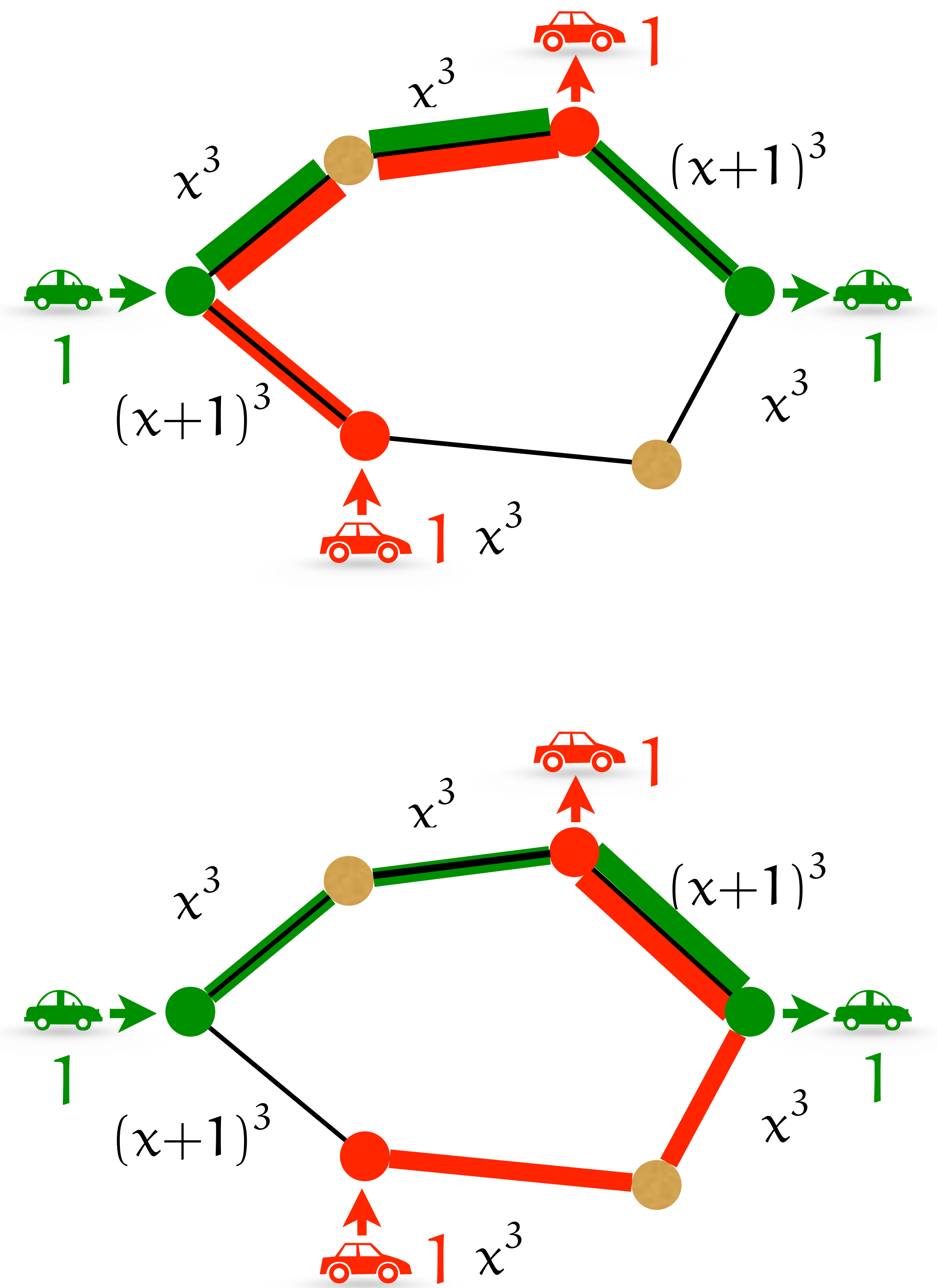


min-cost flow problem
(all edges have unit capacity)

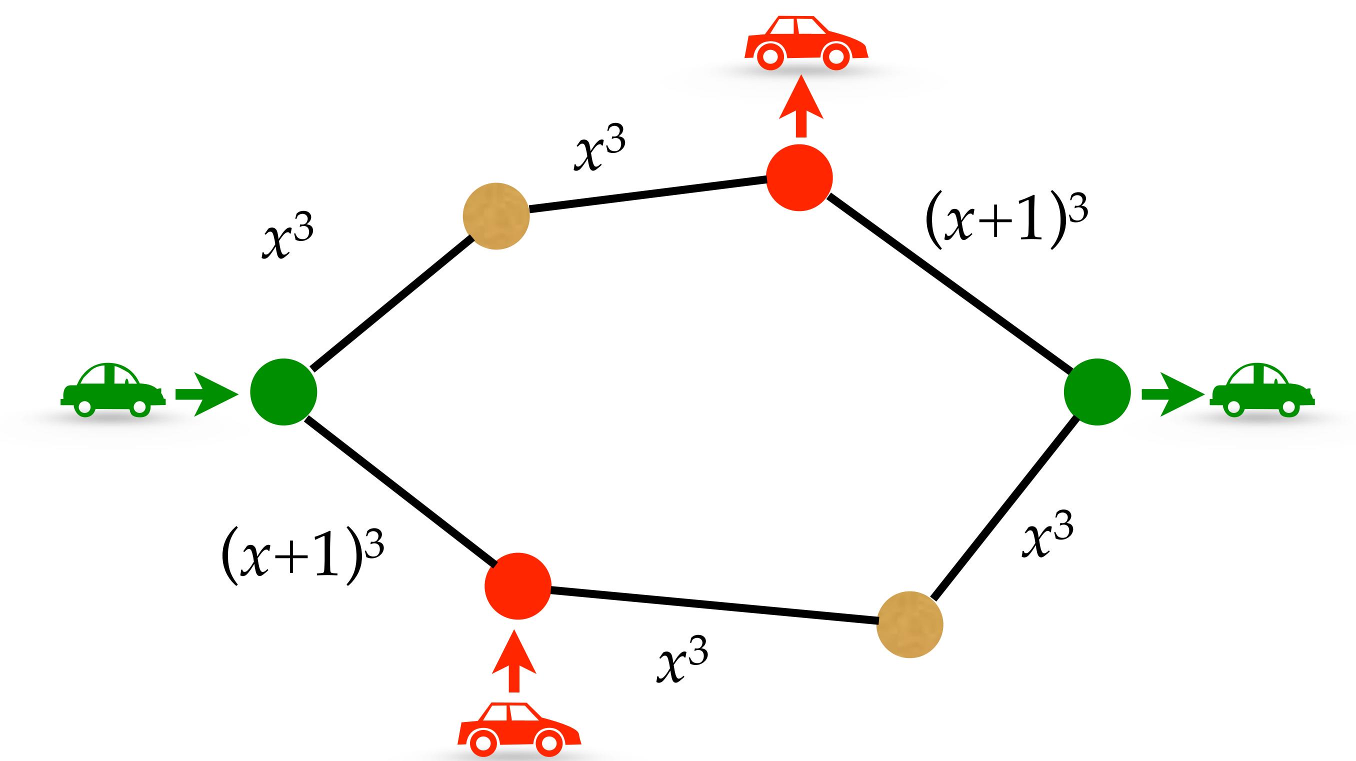
- no positive result for more sources and destinations known (also no result for mixed equilibria)

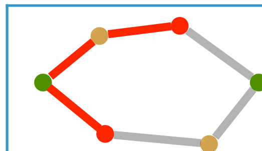
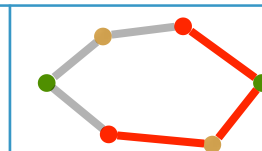
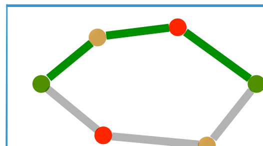
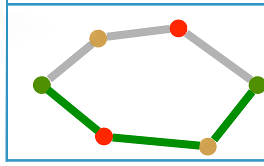
Conclusion for unweighted games

- for unweighted games with unsplittable flow, i.e, $d_i = 1 \ \forall i \in N$ a pure Nash equilibrium always exists
- any sequence of unilateral (single-player) improvements converges to a pure Nash equilibrium
- Nash equilibria are not unique
- computation is in general hard (even two players and affine costs)
- efficient algorithms only known for special cases:
  - single source or single sink
  - Matroids

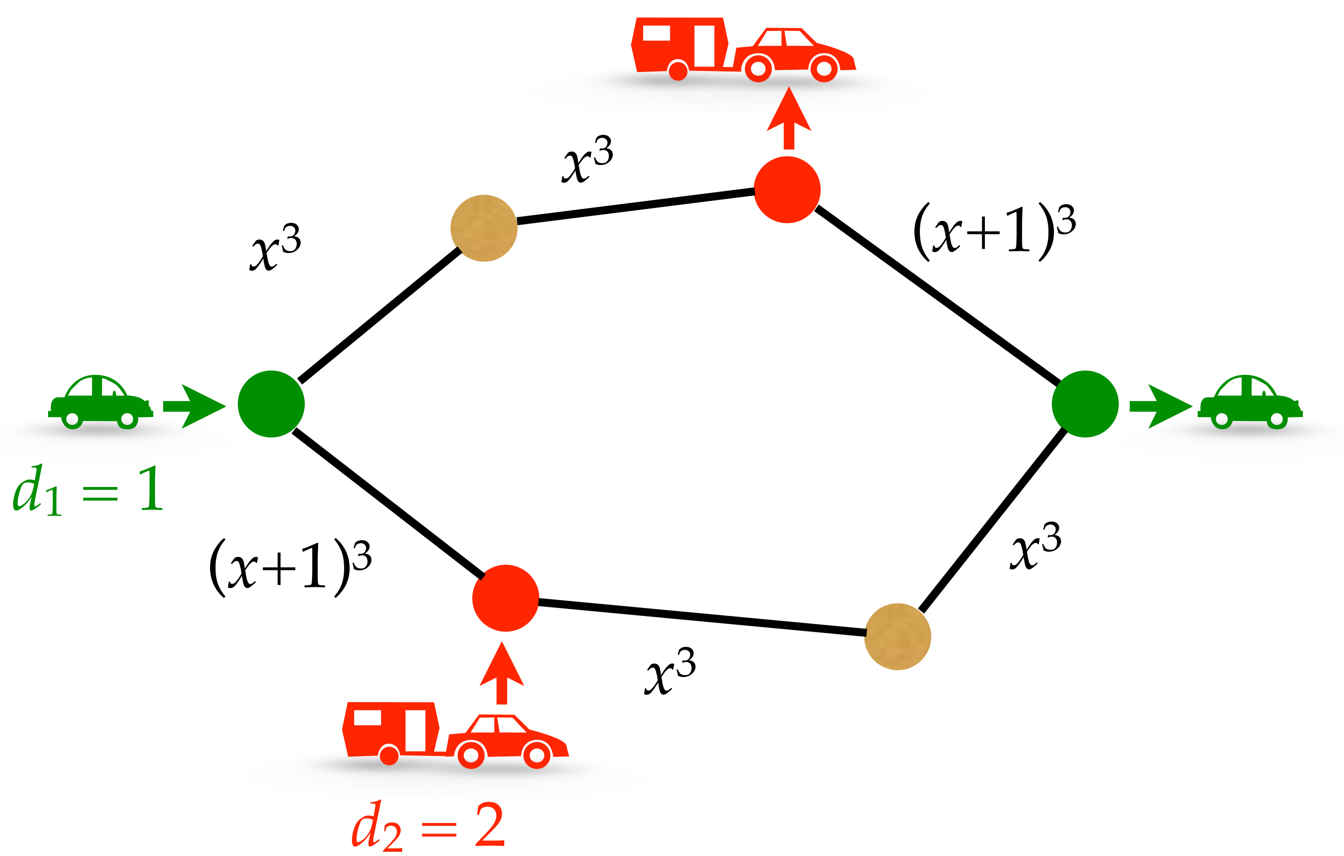


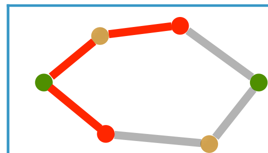
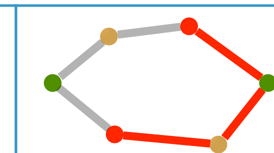
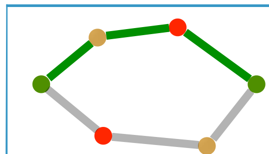
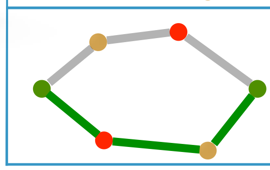
Existence of Nash equilibria



		
	24, 24	29, 29
	29, 29	24, 24

Existence of Nash equilibria

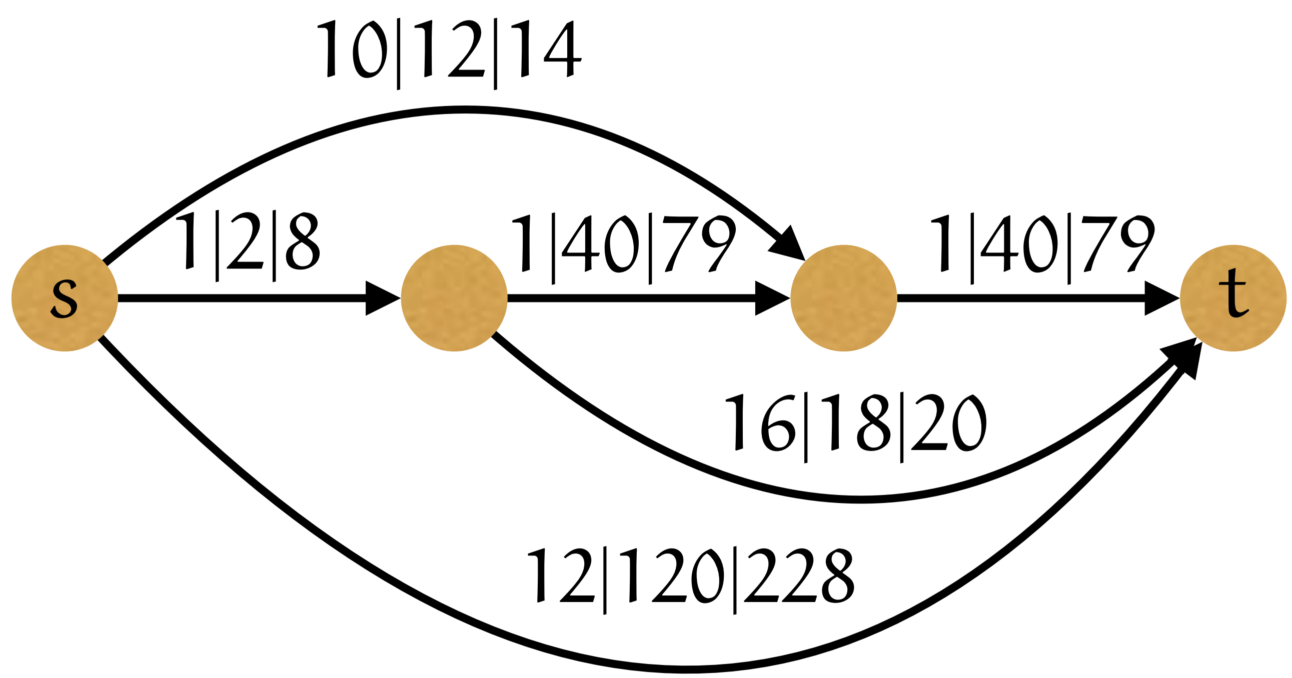


		
	62, 162	66, 160
	66, 160	62, 162

Further counterexamples

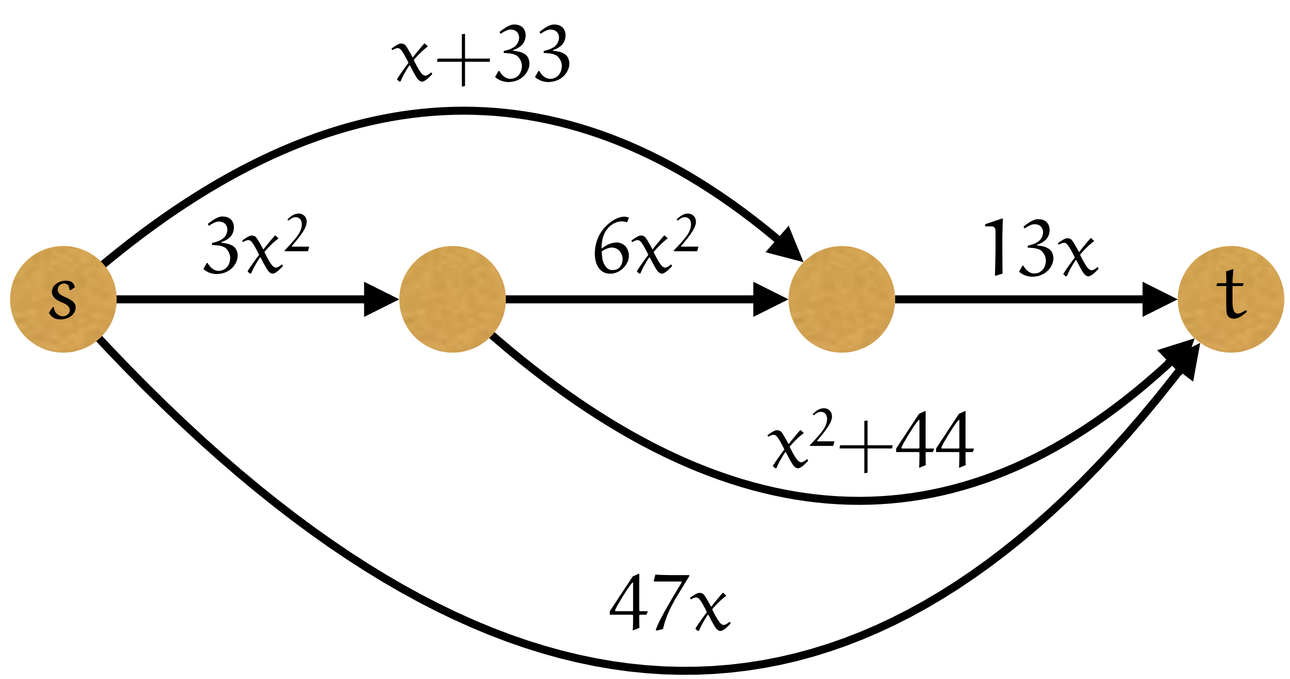
[Fotakis et al. '05]

$d_1 = 1$
$d_2 = 2$



[Goemans et al. '05]

$d_1 = 1$
$d_2 = 2$



Positive results

- Restrictions on the strategy space: A Nash equilibrium exists, if costs are non-decreasing and all strategy spaces $\mathcal{P}_i$ are…
  - singletons, i.e. $|P| = 1$ for all $P \in \mathcal{P}_i$, $i \in N$. [Ieong et al. '05]
  - the basis of a matroid. [Ackermann et al. '09]

- Restrictions on the cost functions: A Nash equilibrium exists, if all cost functions are…
  - affine. →Exercise session [Fotakis et al. '05]
  - of type $c_e(x) = \exp(x)$. [Panagopoulou, Spirakis '06]
  - of type $c_e(x) = k_e / x$ with $b_e \in \mathbb{R}_+$ (for 2-player games). [Anshelevich et al. '08]

Consistent cost functions

Definition — Consistent cost functions

Set of cost functions $\mathscr{C}$, such that all weighted congestion games with costs in $\mathscr{C}$ have a Nash equilibrium.

- $\mathcal{C} = \{c : c(x) = ax + b;\ a, b \in \mathbb{R}\}$ is consistent. [Fotakis et al. '05]
- $\mathcal{C} = \{c : c(x) = \exp(x)\}$ is consistent. [Panagopoulou, Spirakis '06]
- $\mathcal{C} = \{c : c(x) = k_e \,/\, x,\ k_e \in \mathbb{R}_+\}$ is consistent for 2-player games. [Anshelevich et al. '08]

Which are the maximal sets of consistent cost functions?

Characterization for 2-player games

Theorem [Harks, K., '12]

$\mathscr{C}$ is consistent for weighted congestion games with 2 players
if and only if

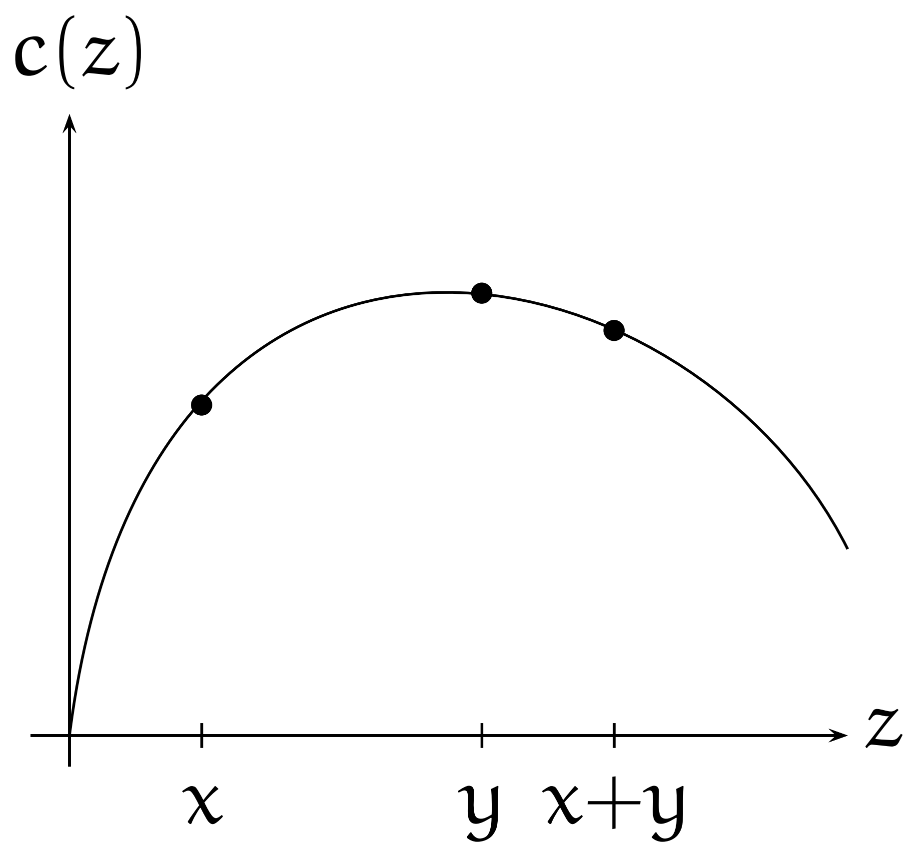
1. $\mathscr{C}$ contains only monotonic functions, and
2. for all $c_1, c_2 \in \mathcal{C}$ there are $a, b \in \mathbb{R}$ with $c_1(x) = a\,c_2(x) + b$.

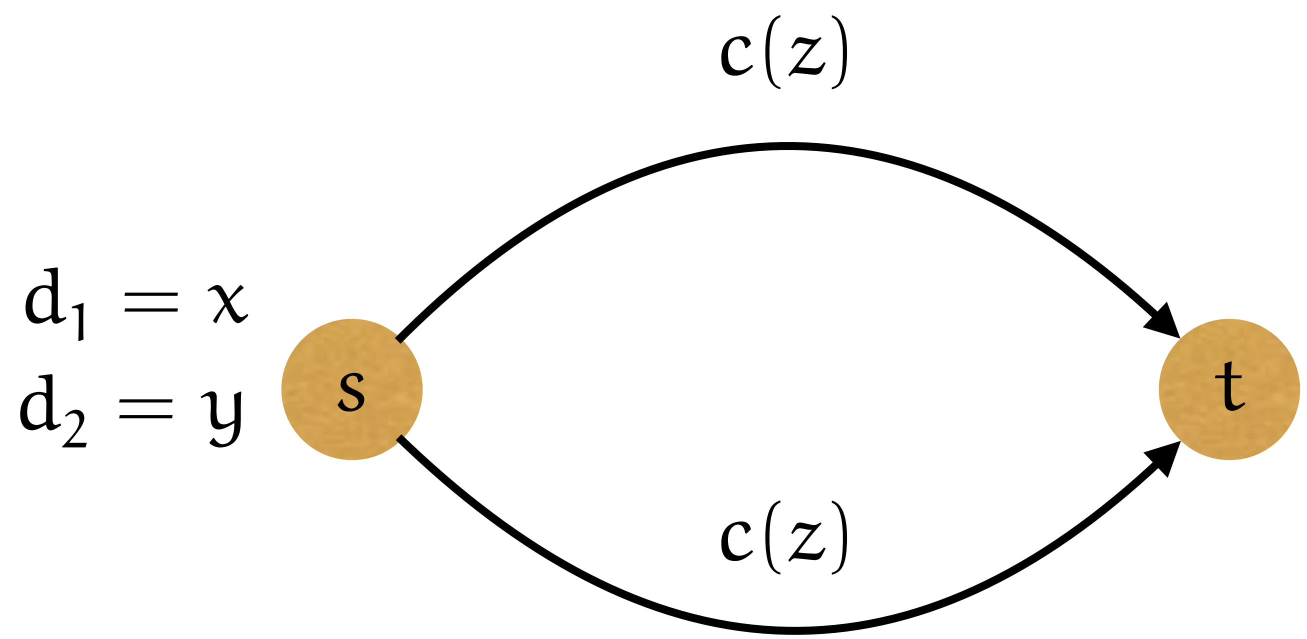
- Assumption: $\mathscr{C}$ contains only continuous functions
- Sufficiency by potential function [Harks, K., Möhring, `11]

Proof "$\Rightarrow$"

- let $\mathscr{C}$ be a set of consistent cost functions.

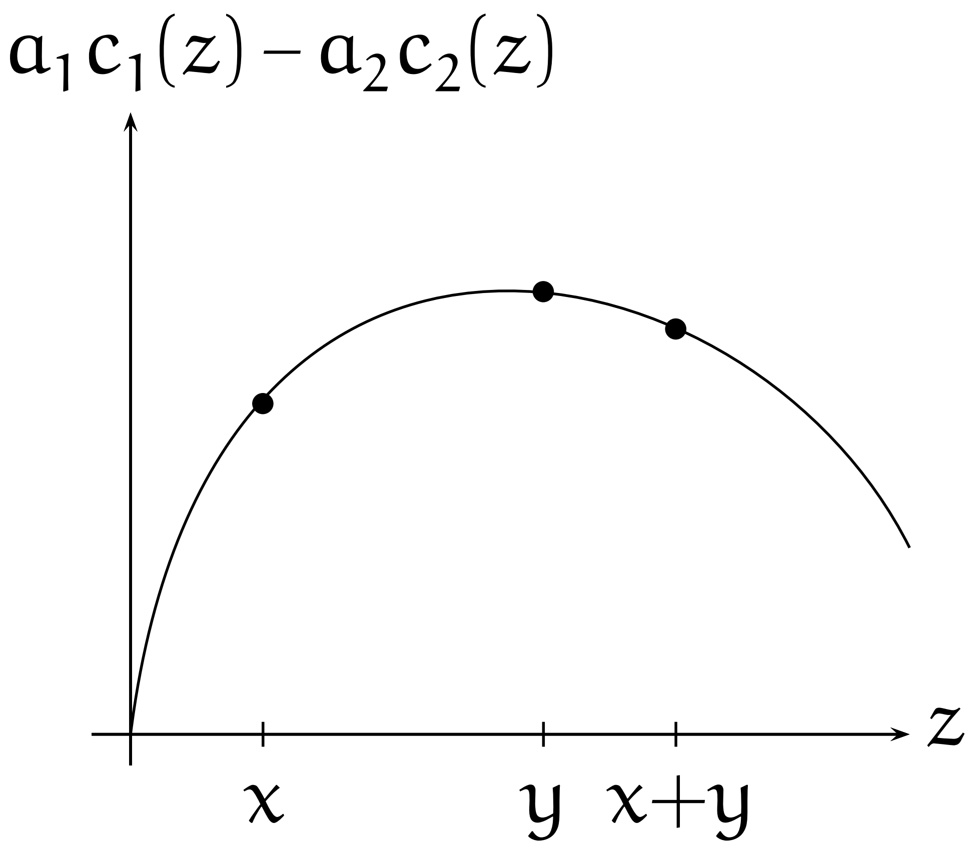
1. Step: Every $c \in \mathscr{C}$ is monotonic.

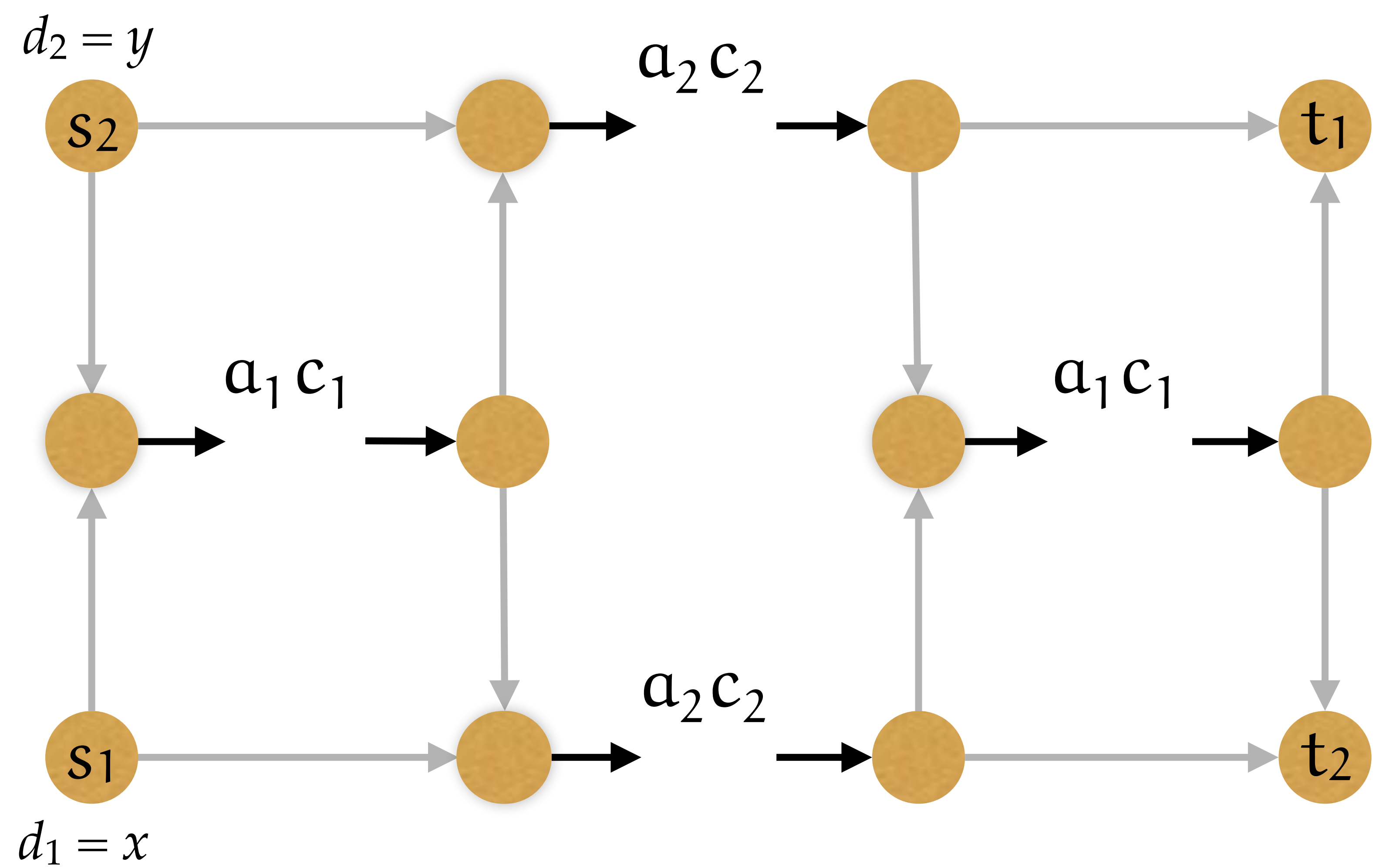




Proof "⇒"

2. Step: $a_1\, c_1(x) - a_2\, c_2(x)$ monotonic for all $a_1, a_2 \in \mathbb{Z}$, $c_1, c_2 \in \mathcal{C}$.





$$a_1c_1(x) - a_2c_2(x) < a_1c_1(x+y) - a_2c_2(x+y) < a_1c_1(y) - a_2c_2(y)$$

$$\Rightarrow\ a_1c_1(x) + a_2c_2(x+y) < a_1c_1(x+y) + a_2c_2(x)$$

$$\Rightarrow\ a_1c_1(y) + a_2c_2(x+y) > a_1c_1(x+y) + a_2c_2(y)$$

Proof "$\Rightarrow$"

3. Step: $a_1 c_1 - a_2 c_2$ monotonic $\forall\, a_1, a_2 \in \mathbb{Z} \Rightarrow \exists\, a, b \in \mathbb{R} : c_1 = a\, c_2 + b$.

- Intuition for twice differentiable functions $c_1, c_2 \in \mathcal{C}$ with $c_1', c_2', c_1'', c_2'' > 0$
  - For a contradiction, assume

    $\nexists\, a, b \in \mathbb{R} : c_1(x) = a\, c_2(x) + b$ for all $x \geqslant 0$
  - $\exists\, x_0,\ \varepsilon > 0 : c_1'(x)/c_2'(x) \neq 0$ for all $x \in (x_0 - \varepsilon,\ x_0 + \varepsilon)$
  - $\det \begin{bmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{bmatrix} \neq 0$ for all $x \in (x_0 - \varepsilon,\ x_0 + \varepsilon)$
  - $\exists\, a_1,\ a_2 :$

    $a_1\ c_1'(x)\ -\ a_2\ c_2'(x) = 0$

    $a_1\ c_1''(x)\ -\ a_2\ c_2''\ (x) \neq 0$

    for some $x \in (x_0 - \varepsilon,\ x_0 + \varepsilon)$
- $a_1 c_1\ -\ a_2 c_2$ has strict extremum in $(x_0 - \varepsilon,\ x_0 + \varepsilon)$.

Characterization for n players

Theorem [Harks, K. '12]

$\mathscr{C}$ is consistent for weighted congestion games <u>if and only if</u>

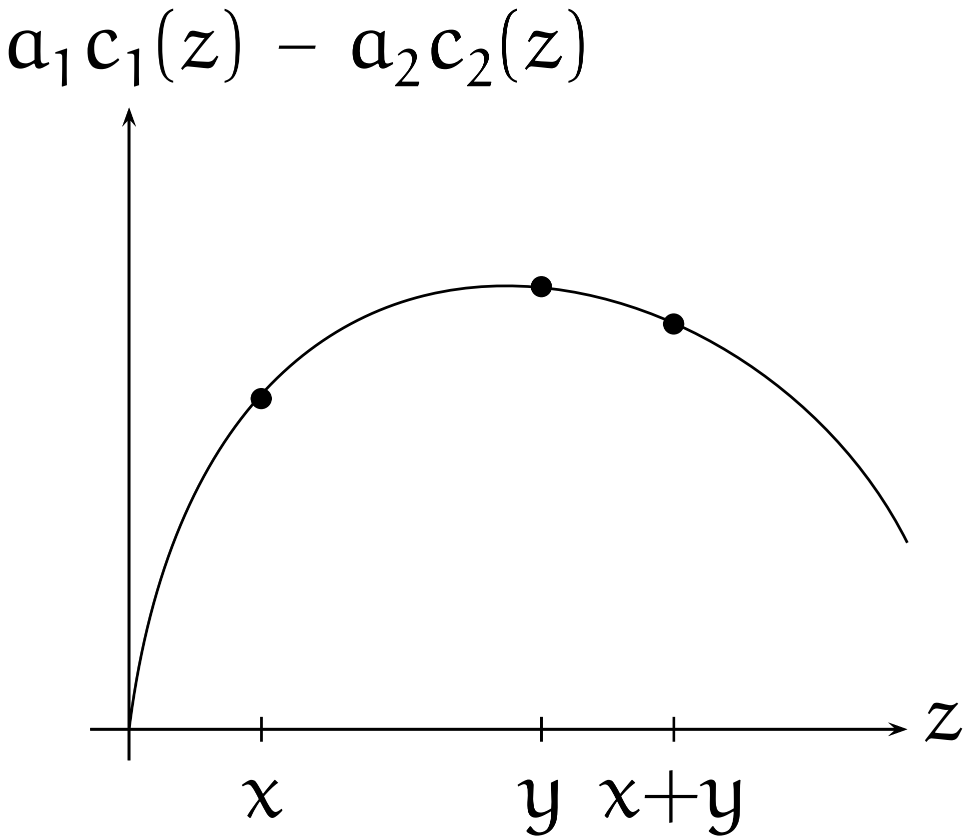
1. $\mathscr{C}$ only contains affine functions of type $ax + b$, <u>or</u>
2. $\mathscr{C}$ only contains exponential functions of type $a\exp(\varphi x) + b$,

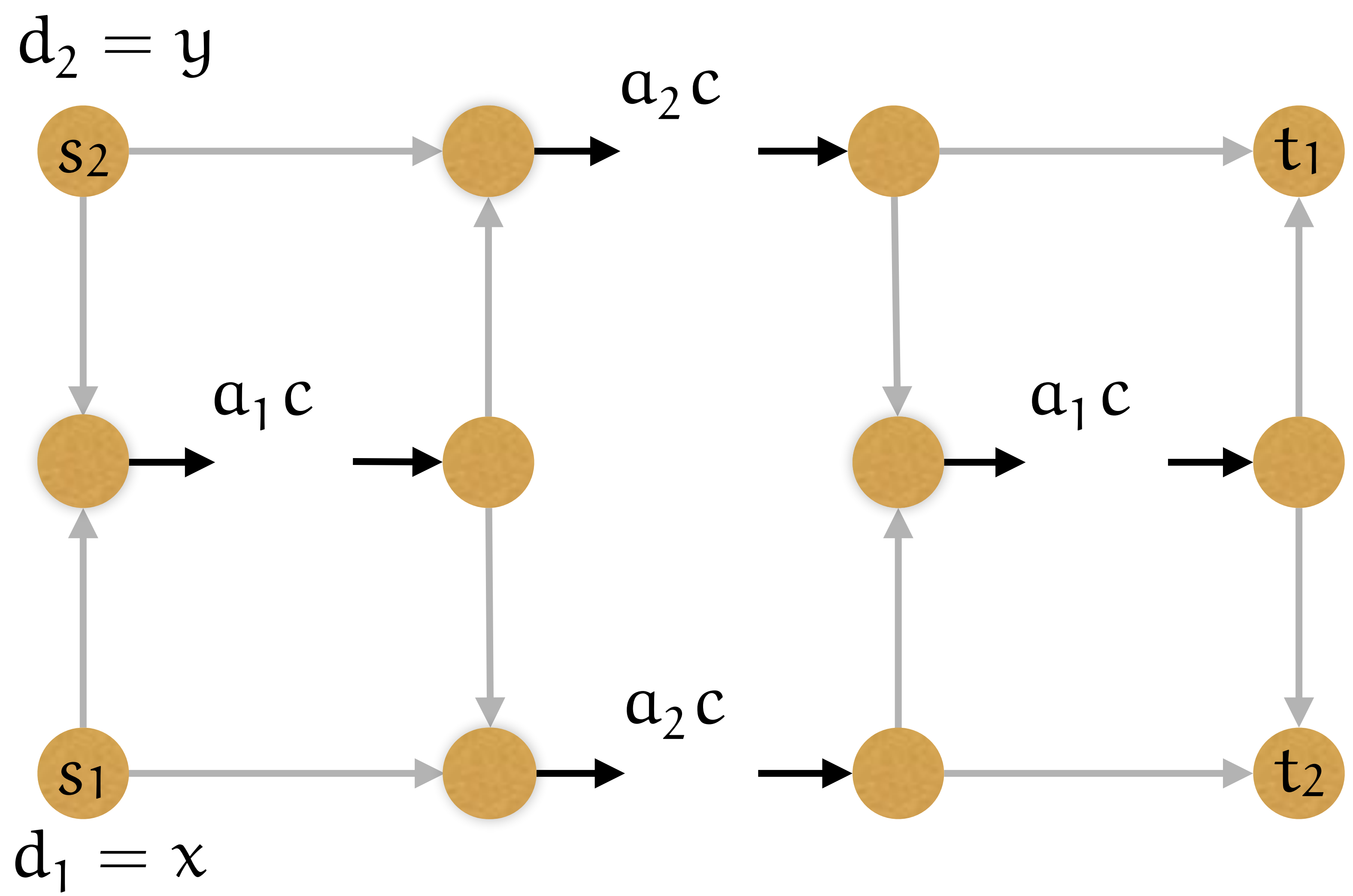
   where $a, b \in \mathbb{R}$ may depend on c, while φ is independent of c.

- Assumption: $\mathscr{C}$ only contains continuous functions
- Sufficiency of conditions follows from [Harks, K., Möhring, '11]

Proof "$\Rightarrow$"

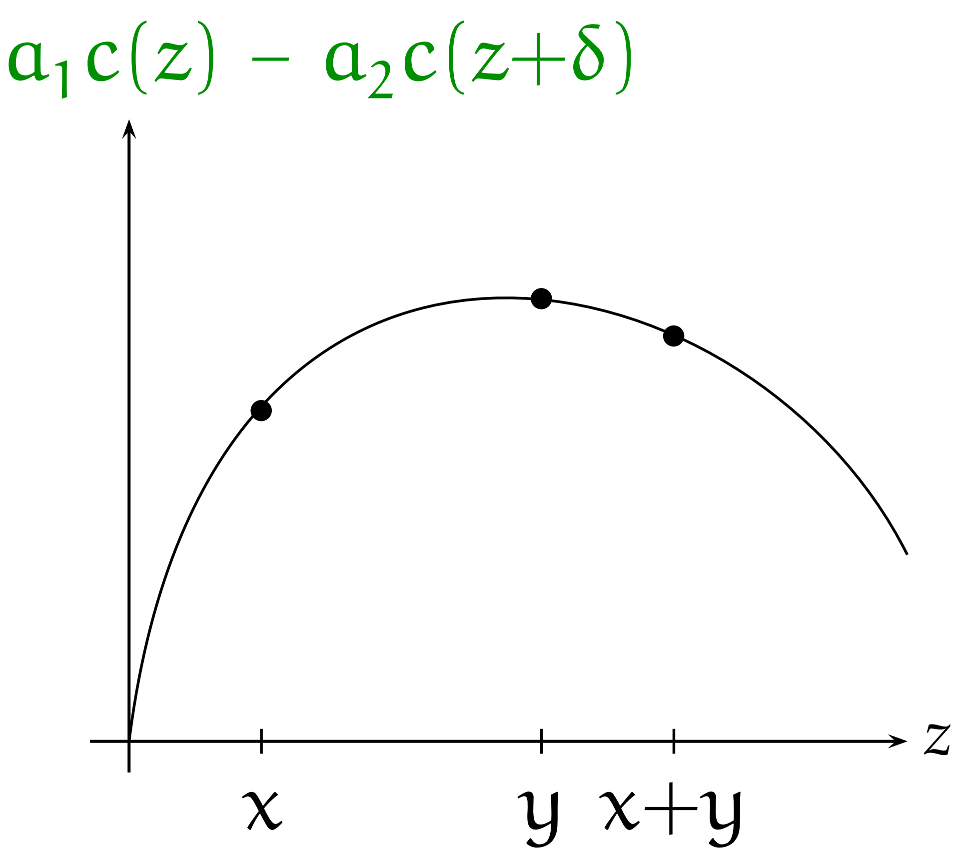
1. Step: $c \in \mathcal{C} \Rightarrow a_1 c(x) - a_2 c(x+\delta)$ monotonic for all $a_1, a_2 \in \mathbb{Z}$, $\delta > 0$.

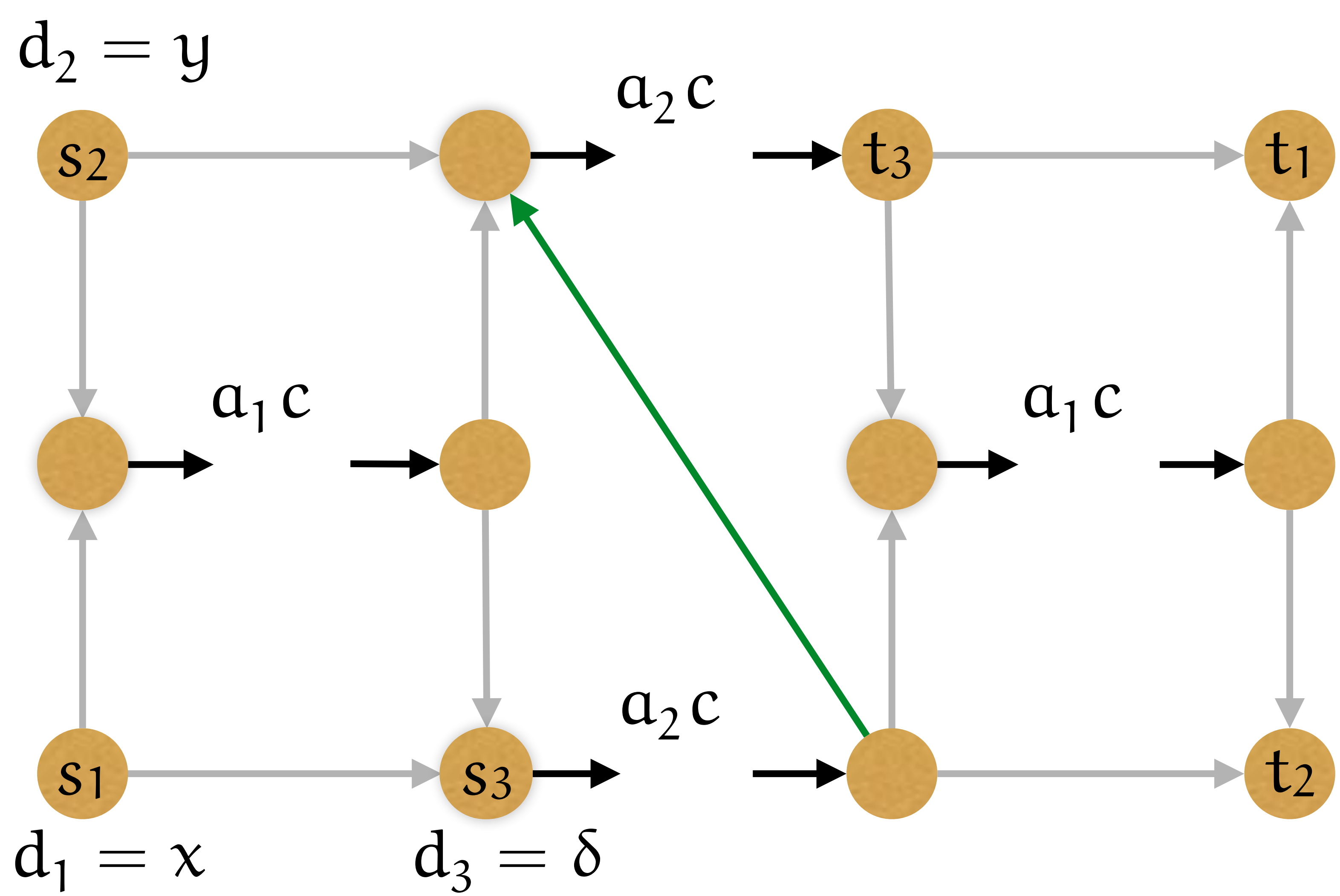




Proof "$\Rightarrow$"

1. Step: $c \in \mathcal{C} \Rightarrow a_1 c(x) - a_2 c(x+\delta)$ monotonic for all $a_1, a_2 \in \mathbb{Z}$, $\delta > 0$.





- for all $\delta > 0$ there are $a, b \in \mathbb{R}$, with $c(x+\delta) = a\,c(x) + b$.

Proof "$\Rightarrow$"

- for all $\delta > 0$ there are $a, b \in \mathbb{R}$ with $c(x+\delta) = a\,c(x) + b$, i.e,

$$\begin{aligned}
c(1\delta) &= a\,c(0\delta) + b \\
c(2\delta) &= a\,c(1\delta) + b \\
&\dots \\
c((k+1)\delta) &= a\,c(k\delta) + b \\
c((k+2)\delta) &= a\,c((k+1)\delta) + b
\end{aligned}$$

- $0 = c((k+2)\delta) - (a+1)\,c((k+1)\delta) + a\,c(k\delta)$ for all $k \in \mathbb{N}$.
- solution of the linear recurrence relation
  - if $a \neq 1$: $c(k\delta) = \beta + \alpha\, a^k = \alpha \exp(k \ln |a|) + \beta$.
  - if $a = 1$: $c(k\delta) = \beta + \alpha\, k$.

Further characterizations

Uniform, variable demands [Harks, K. '15]

- $\pi_i(s) = U_i(d_i) - \sum_{r \in s_i} c_r(x_r(s))$,
- $x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_i$, $S_i \subseteq 2^R \times \mathbb{R}$

homogeneously exponential functions

Variable demands [Harks, K. '15]

- $\pi_i(s) = U_i(d_i) - \sum_{r \in s_i} d_i\, c_r(x_r(s))$,
- $x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_i$, $S_i \subseteq 2^R \times \mathbb{R}$

affine functions
or
homogeneously exponential functions

Uniform, res.-dep. demands [Harks, K. '12]

- $\pi_i(s) = \sum_{r \in s_i} c_r(x_r(s))$,
- $x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_{i,r}$

contant functions

Resource-dep. demands [Harks, K. '12]

- $\pi_i(s) = \sum_{r \in s_i} d_{i,r}\, c_r(x_r(s))$,
- $x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_{i,r}$

affine functions

Uniform, weighted [Harks, K. '12]

- $\pi_i(s) = \sum_{r \in s_i} c_r(x_r(s))$,
- $x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_i$

affine functions
or
exponential functions

Weighted [Harks, K. '12]

$\pi_i(s) = \sum_{r \in s_i} d_i\, c_r(x_r(s))$,

$x_r(s) = \sum_{i \in N \,:\, r \in s_i} d_i$

affine functions
or
exponential functions

Unweighted [Rosenthal '73]

- $\pi_i(s) = \sum_{r \in s_i} c_r(x_r(s))$,
- $x_r(s) = |\, i \in N : r \in s_i|$

all functions

Conclusion

- existence:
  - equilibria exist for non-atomic players
  - equilibria exist for unweighted atomic players
  - equilibria may not exist for weighted atomic players (only for affine or exponential cost functions)
- computation:
  - convex programming for non-atomic players
  - efficient only for special cases (single source, Matroid) for unweighted players
  - open for weighted atomic players
- efficiency:
  - Wardrop equilibria generalize electric networks which minimize energy dissipation
  - general road networks are not efficient wrt total travel time
  - inefficiency can be bounded in terms of the price of anarchy